Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 765–768
COMPTES RENDUS
MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

A Bismut type theorem for subelliptic heat semigroups

Fabrice Baudoin

Institut de mathématiques de Toulouse, Université Paul-Sabatier, 118, route de Narbonne, 31062 Toulouse cedex 9, France

Received 6 February 2007; accepted after revision 15 May 2007

Presented by Paul Malliavin

Abstract

Given a general second order subelliptic differential operator $\mathcal{L}$ defined on a vector bundle $\mathcal{E}$ over a compact manifold, we study the existence of $\lim_{t\to 0}\sigma(p_t(x,x))$, where p_t is the heat kernel of $e^{t\mathcal{L}}$ and σ is a linear map on $\mathbf{End}(\mathcal{E}_x)$. Our result contains as a special case the local Atiyah–Singer index theorem for Dirac operators on Clifford bundles. Our approach is based on an extension to fiber bundles of the link pointed out by Rotschild and Stein between Nilpotent Lie groups and subelliptic heat kernel asymptotics on the diagonal. ***To cite this article: F. Baudoin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Un Théorème de type Bismut pour les semi-groupes de la chaleur sous-elliptiques. Étant donné un opérateur sous-elliptique $\mathcal{L}$ défini sur un fibré vectoriel $\mathcal{E}$ au dessus d'une variété riemannienne compacte, nous étudions l'existence de $\lim_{t\to 0}\sigma(p_t(x,x))$, où p_t est le noyau de la chaleur du semi-groupe $e^{t\mathcal{L}}$ et où σ is a linear map on $\mathbf{End}(\mathcal{E}_x)$ est une application linéaire sur $\mathbf{End}(\mathcal{E}_x)$. Notre résultat contient en particulier le théorème de l'indice local d'Atiyah–Singer pour les opérateurs de Dirac sur les fibrés de Clifford. Notre approche repose sur une extension aux fibrés vectoriels du lien mis en avant par Rotschild et Stein qui existe entre les groupes de Lie nilpotents et l'asymptotique sur la diagonale d'un noyau de la chaleur sous-elliptique. ***Pour citer cet article : F. Baudoin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Preliminaries: Carnot groups, paths integrals and strongly regular points for subelliptic operators

1.1. Carnot groups

Definition 1.1. A Carnot group of step N is a simply connected Lie group $\mathbb{G}$ whose Lie algebra can be written $\mathfrak{g}=\mathcal{V}_1\oplus\cdots\oplus\mathcal{V}_N$, where $[\mathcal{V}_i,\mathcal{V}_j]\subset\mathcal{V}_{i+j}$ and $\mathcal{V}_s=0$, for $s>N$.

On $\mathfrak{g}$ we can consider the family of linear operators $\delta_t:\mathfrak{g}\to\mathfrak{g}$, $t\geqslant 0$ which act by scalar multiplication t^i on $\mathcal{V}_i$. These operators are Lie algebra automorphisms due to the grading. The maps δ_t induce Lie group automorphisms $\Delta_t:\mathbb{G}\to\mathbb{G}$ which are called the canonical dilations of $\mathbb{G}$. The number

E-mail address: fbaudoin@cict.fr.

1631-073X/$ – see front matter © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2007.05.005

$$D = \sum_{i=1}^{N} i \dim \mathcal{V}_i$$

is called the homogeneous dimension of $\mathbb{G}$.

Definition 1.2. A Lie group morphism $\psi : \mathbb{G}_1 \to \mathbb{G}_2$ is called a Carnot group morphism if

$$\psi \circ \Delta_t^1 = \Delta_t^2 \circ \psi, \quad t \geqslant 0,$$

where Δ_t^1 (resp. Δ_t^2) are the canonical dilations of $\mathbb{G}^1$ (resp. $\mathbb{G}_2$).

1.2. Carnot groups and path integrals

Let $\mathbb{R}[[X_0, \ldots, X_d]]$ be the non commutative algebra over $\mathbb{R}$ of the formal series with $d+1$ indeterminates, that is the set of series

$$Y = \sum_{k \geqslant 0} \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}.$$

The bracket between two elements U and V of $\mathbb{R}[[X_0, \ldots, X_d]]$ is given by $[U, V] = UV - VU$.

If $I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ is a word, we denote $d(I) = n(I) + k$, where $n(I)$ is the number of 0 in I and by X_I the commutator defined by $X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$.

For $N \geqslant 1$, let us denote $\mathfrak{g}_{N,d}$ the Lie algebra $\mathbb{R}[[X_0, \ldots, X_d]]$ quotiented by the relations

$$\{X_I = 0, d(I) \geqslant N + 1\},$$

and $\pi : \mathbb{R}[[X_0, \ldots, X_d]] \to \mathfrak{g}_{N,d}$ the canonical surjection. We can write the stratification $\mathfrak{g}_N = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N$, where

$$\mathcal{V}_k = \mathbf{span}\{\pi(X_I), d(I) = k\}.$$

The set $\mathfrak{g}_{N,d}$ endowed with the group law given by the Baker–Campbell–Hausdorff formula is a Carnot group that will be denoted $\mathbb{G}_{N,d}$.

We denote by $\mathcal{S}_k$ the set of the permutations of $\{0, \ldots, k\}$. If $\sigma \in \mathcal{S}_k$, we denote $e(\sigma)$ the cardinality of the set $\{j \in \{0, \ldots, k-1\}, \sigma(j) > \sigma(j+1)\}$, and $\sigma(I)$ the word $(i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.

If $x : [0, 1] \to \mathbb{R}^d$ is an absolutely continuous path and $I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ a word, we use the notation,

$$\Lambda_I(x)_t = \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t} \mathrm{d}x_{t_1}^{\sigma^{-1}(i_1)} \cdots \mathrm{d}x_{t_k}^{\sigma^{-1}(i_k)},$$

with the convention that $x_t^0 = t$.

Definition 1.3. If $x : [0, 1] \to \mathbb{R}^d$ is an absolutely continuous path,

$$x_t^N = \sum_{I, d(I) \leqslant N} \Lambda_I(x)_t \pi(X_I)$$

is called the lift of x in $\mathbb{G}_{N,d}$.

Remark 1.4. The previous definition can easily be extended to Brownian paths by replacing in the definition of $\Lambda_I(x)_t$ Riemann–Stieltjes integrals by Stratonovitch integrals.

1.3. Strongly regular points for subelliptic operators

Let $\mathbb{M}$ be a n-dimensional compact smooth manifold and let $\mathcal{L}$ be an operator on $\mathbb{M}$ that can be written

$$\mathcal{L} = V_0 + \sum_{i=1}^{d} V_i^2,$$

the V_i's being smooth vector fields.

For $x_0 \in \mathbb{M}$, and $N \geqslant 1$ let us consider the smooth map

$$\pi_{x_0}^N : \mathbb{G}_{N,d} \to \mathbb{M}$$

such that for any absolutely continuous path $y : [0,1] \to \mathbb{R}^d$

$$\pi_{x_0}^N\big(y_1^N\big) = \exp\bigg(\sum_{I,d(I)\leqslant N} \Lambda_I(y)_1 V_I\bigg)(x_0),$$

where y^N is the lift of y in $\mathbb{G}_{N,d}$. According to Chow's theorem, such a map is well-defined in a unique way.

Definition 1.5. We say that $x_0 \in \mathbb{M}$ is strongly regular for $\mathcal{L}$ if there exist

(i) A Carnot group $\mathbb{G}(x_0)$;
(ii) A local diffeomorphism $\Phi_{x_0} : \mathcal{O} \to \mathcal{O}'$ ($\mathcal{O}$ is a neighborhood of 0 in $\mathbb{G}(x_0)$ and $\mathcal{O}'$ a neighborhood of x_0 in $\mathbb{M}$);
(iii) A surjective Carnot group morphism $\Psi : \mathbb{G}_N \to \mathbb{G}(x_0)$, for some $N \geqslant 1$; such that $\pi_{x_0}^N = \Phi_{x_0} \circ \Psi$.

Remark 1.6. If $x_0 \in \mathbb{M}$ is a strongly regular point then Hörmander's condition is obviously satisfied at x_0, that is

$$\mathbf{dim}\,\mathbf{span}\big\{V_I(x_0), d(I) \leqslant N\big\} = \mathbf{dim}\,\mathbb{M}.$$

Proposition 1.7. *If $x_0 \in \mathbb{M}$ is a strongly regular point then, up to isomorphism, the Carnot group $\mathbb{G}(x_0)$ is unique, it shall be called the tangent space to $\mathcal{L}$ at x_0.*

Example 1.8 *(Maximal subellipticity).* For instance, if

$$\mathbf{dim}\,\mathbf{span}\big\{V_I(x_0), d(I) \leqslant N\big\} = \mathbf{dim}\,\mathbb{M} = \mathbf{dim}\,\mathbb{G}_{N,d},$$

then, it is easily seen that x_0 is a strongly regular point with tangent space $\mathbb{G}_{N,d}$.

2. A Bismut type theorem for subelliptic heat semigroups

Let $\mathbb{M}$ be a compact smooth Riemannian manifold and let $\mathcal{E}$ be a finite-dimensional vector bundle over $\mathbb{M}$. We denote by $\Gamma(\mathbb{M}, \mathcal{E})$ the space of smooth sections. Let now ∇ denote a connection on $\mathcal{E}$. We consider the following linear partial differential equation

$$\frac{\partial \Phi}{\partial t} = \mathcal{L}\Phi, \quad \Phi(0,x) = f(x), \tag{1}$$

where $\mathcal{L}$ is an operator on $\mathcal{E}$ that can be written $\mathcal{L} = \nabla_0 + \sum_{i=1}^d \nabla_i^2$, with $\nabla_i = \mathcal{F}_i + \nabla_{V_i}$, $0 \leqslant i \leqslant d$, the V_i's being smooth vector fields on $\mathbb{M}$ and the $\mathcal{F}_i$'s being smooth potentials (that is sections of the bundle $\mathbf{End}(\mathcal{E})$). It is known that the solution of (1) can be written

$$\Phi(t,x) = \big(\mathrm{e}^{t\mathcal{L}} f\big)(x) = \mathbf{P}_t f(x).$$

Let us assume that $x_0 \in \mathbb{M}$ is a strongly regular point of $V_0 + \sum_{i=1}^d V_i^2$ with tangent space $\mathbb{G}(x_0)$ whose homogeneous dimension is denoted D. From Hörmander's theorem, there exists a smooth map $p(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma(\mathbb{M}, \mathbf{End}(\mathcal{E}))$ such that for $\eta \in \Gamma(\mathbb{M}, \mathcal{E})$,

$$(\mathbf{P}_t \eta)(x_0) = \int_{\mathbb{M}} p_t(x_0, y)\eta(y)\,\mathrm{d}y.$$

If $I \in \{0,1,\ldots,d\}^k$ is a word, we denote $\nabla_I = [\nabla_{i_1}, [\nabla_{i_2}, \ldots, [\nabla_{i_{k-1}}, \nabla_{i_k}]\ldots]$, and $\mathcal{F}_I = \nabla_I - \nabla_{V_I} \in \Gamma(\mathbb{M}, \mathbf{End}(\mathcal{E}))$. If $I \in \{0,1,\ldots,d\}^k$ is a word, we still denote $\nabla_I = [\nabla_{i_1}, [\nabla_{i_2}, \ldots, [\nabla_{i_{k-1}}, \nabla_{i_k}]\ldots]$, and $\mathcal{F}_I = \nabla_I - \nabla_{V_I} \in \Gamma(\mathbb{M}, \mathbf{End}(\mathcal{E}))$.

For $t>0$, let us consider the operator Θ_t^N defined on $\Gamma(\mathbb{M},\mathcal{E})$ by the property that for $\eta\in\Gamma(\mathbb{M},\mathcal{E})$ and $y\in\mathcal{O}_{x_0}$,

$$\left(\Theta_t^N\eta\right)(y)=\mathbb{E}\left(\left[\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\nabla_I\right)\eta\right](x_0)\,\Big|\,\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)=0\right),$$

where $(B_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion.

Lemma 2.1. *For $t>0$, Θ_t^N is a smooth section of the bundle* $\mathbf{End}(\mathcal{E})$.

Proposition 2.2. *Let $r(x_0)$ be the degree of non holonomy at x_0 (see* [1] *pp.* 61*). For $N\geqslant r(x_0)$, when $t\to 0$,*

$$p_t(x_0,x_0)=q_t^N(x_0)\Theta_t^N(x_0)+\mathrm{O}\left(t^{\frac{N+1-D}{2}}\right),$$

where $q_t^N(x_0)$ is the density at 0 *of the random variable $\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)$.*

Our main theorem is the following:

Theorem 2.3. *Assume that σ_{x_0} is a linear map on $\mathbf{End}(\mathcal{E}_{x_0})$ such that $\sigma_{x_0}(\mathcal{F}_{I_1}(x_0)\cdots\mathcal{F}_{I_k}(x_0))=0$ if $\sum_{j=1}^k d(I_j)<D$. Then we have*

$$\lim_{t\to 0}\sigma_{x_0}\left(p_t(x_0,x_0)\right)=\sum C_{I_1,\dots,I_k}\sigma_{x_0}\left(\mathcal{F}_{I_1}(x_0)\cdots\mathcal{F}_{I_k}(x_0)\right),$$

for some universal constants $C_{I_1,\dots,I_k}$, where the above sum is taken over all the words $I_1,\dots,I_k$ such that $\sum_{j=1}^k d(I_j)=D$.

In the elliptic case, previous corollary leads to a new proof of Atiyah–Singer local index theorem. Indeed, let us here, assume that $\mathbb{M}$ admits a spin structure. The spin bundle $\mathcal{S}$ over $\mathbb{M}$ is the vector bundle such that for every $x\in\mathbb{M}$, $\mathcal{S}_x$ is the spinor module over the cotangent space $\mathbf{T}_x^*\mathbb{M}$. At each point x, there is therefore a natural action of the Clifford algebra $\mathbf{Cl}(\mathbf{T}_x^*\mathbb{M})\simeq\mathbf{End}(\mathcal{S}_x)$; this action will be denoted by $\mathbf{c}$. On $\mathcal{S}$, there is a canonical elliptic first-order differential operator called the Dirac operator and denoted $\mathbf{D}$. In a local orthonormal frame e_i, with dual frame e_i^*, we have $\mathbf{D}=\sum_i c(e_i^*)\nabla_{e_i}$, where ∇ is the Levi-Civita connection. By applying the above theorem to $\mathcal{L}=-\mathbf{D}^2$ and $\sigma=\mathbf{Str}$, where $\mathbf{Str}$ is the supertrace, we obtain:

Proposition 2.4. *(See [2]) For $x\in\mathbb{M}$,*

$$\lim_{t\to 0}\mathbf{Str}\,p_t(x,x)=\frac{1}{(4\pi)^{d/2}(d/2)!}\mathbf{Str}\,\mathbb{E}\left(\left(\sum_{1\leqslant i<j\leqslant d}DR(e_i,e_j)\int_0^1 B_s^i\,\mathrm{d}B_s^j-B_s^j\,\mathrm{d}B_s^i\right)^{d/2}\,\Big|\,B_1=0\right),$$

where p_t is the heat kernel of $\mathrm{e}^{-t\mathbf{D}^2}$, *and $DR(e_i,e_j)=\frac{1}{2}\sum_{1\leqslant k<l\leqslant d}\langle R(e_i,e_j)e_k,e_l\rangle e_k^*e_l^*$, with R Riemannian curvature.*

References

[1] F. Baudoin, An Introduction to the Geometry of Stochastic Flows, Imperial College Press, 2004.
[2] J.M. Bismut, The Atiyah–Singer theorems: A probabilistic approach, Part I, J. Func. Anal. 57 (1984), Part II: 57 (1984) 329–348.