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Lyapunov analysis and stabilization to the rest state for solutions to the 1D-barotropic compressible Navier–Stokes equations

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Abstract

In this Note, we establish new estimates for the long time behavior of the solutions to the Navier–Stokes Equations for a compressible barotropic fluid in 1D, with homogeneous Dirichlet boundary conditions, with large initial data, and under the influence of a large mass force in the case when the stationary density admits vacua: a highly singular problem. As a consequence we bring new answers to the question of the stabilizing rate of convergence. ***To cite this article: P. Penel, I. Straškraba, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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Résumé

Analyse de Lyapunov et stabilisation vers l'état d'équilibre pour les solutions des equations de Navier–Stokes compressibles unidimensionnelles. Dans cette Note, nous établissons de nouvelles estimées pour le comportement asymptotique en temps des solutions des équations unidimensionnelles de Navier–Stokes pour un fluide compressible barotropique, associées à des conditions aux limites homogènes de Dirichlet, pour de larges conditions initiales, sous l'influence de larges forces externes telles que la densité stationnaire peut s'annuler : un problème fortement singulier. Comme conséquence nous apportons une réponse nouvelle à la question du taux de convergence. ***Pour citer cet article : P. Penel, I. Straškraba, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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La procédure de construction d'une fonctionnelle de Lyapunov est connue dans le cas de densité stationnaire globalement minorée [8]. En modifiant soigneusement la procédure, on démontre qu'elle est opportune même si l'on perd toute borne inférieure uniforme pour la densité (ce qui est le cas ici puisque l'on s'intéresse à la situation où l'unique équilibre présente un ensemble de mesure nulle où la densité peut s'annuler : un exemple sera donné dans le texte anglais).

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Les résultats obtenus conduisent à une estimation du taux de convergence vers l'équilibre. Ils prennent en compte une classe assez large de fonctions d'état descriptives de la pression, incluant les cas $p(\rho)=\rho^\gamma$ avec $\gamma>1$ quelconque. Ils sont énoncés de façon précise au théorème 7, avec l'inégalité d'énergie généralisée

$$\|\sqrt{\rho}\,u\|_2^2+\|\rho-\rho_\infty\|_\beta^\beta+\|p(\rho)-p(\bar\rho)\|_2^2$$
$$\leqslant c\left\{\mathrm{e}^{-\alpha(t-t_0)}\left(1+\int_{t_0}^{t}\mathrm{e}^{\alpha s}\|g(s,\cdot)\|_2^2\,\mathrm{d}s\right)+\int_t^\infty\|g(s,\cdot)\|_2^2\,\mathrm{d}s\right\}\quad\text{pour tous } t>t_0\geqslant 0.$$

Les notations, les hypothèses et les idées-clés de démonstration seront donnés dans le texte anglais, pour les détails nous renvoyons à l'article [6]. L'introduction d'une densité quasi-stationnaire, notée $\bar\rho$, (voir (11), (12)) (voir aussi [4,8]), avec une contrainte pour la valeur moyenne de $p(\bar\rho)$, nous semble essentielle.

1. Introduction

We deal with the following Navier–Stokes initial-boundary value problem in the domain $Q_T=(0,T)\times(0,l)$, $0<T\leqslant\infty$

$$\rho_t+(\rho u)_x=0\quad\text{in } Q_\infty, \tag{1}$$
$$(\rho u)_t+\left(\rho u^2\right)_x-\left(\mu u_x-p(\rho)\right)_x=\rho f\quad\text{in } Q_\infty, \tag{2}$$
$$u\,|_{x=0,l}=0\quad\text{in } (0,\infty), \tag{3}$$
$$\rho\,|_{t=0}=\rho^0\quad\text{and}\quad u\,|_{t=0}=u^0\quad\text{in } (0,l), \tag{4}$$

u denotes the velocity, ρ the density, $\mu>0$ the viscosity coefficient.

Let the initial functions ρ^0 and u^0 be given in $H^1(0,l))$ and satisfy $\rho^0>0$ and $u^0|_{x=0,l}=0$; $m=\int_0^l\rho^0(x)\,\mathrm{d}x>0$ is assumed to be also given.

Our main requirements on the state function $p(\cdot)$ are as follows:

$$\begin{cases} p(\cdot)\text{ continuous, increasing on } [0,\infty),\ p(0)=0,\ p(\infty)=\infty,\\ p'(\cdot)\in L^\infty_{\mathrm{loc}}(0,\infty),\ p'(r)>0 \text{ when } r>0,\ p(r)\sim r^\gamma \text{ as } r\to 0^+ \text{with a certain } \gamma>0,\\ rp'(r)\leqslant cste \text{ as } r\to 0^+. \end{cases}$$

It is standard that the system (1)–(4) has a strong solution (u,ρ) for any T

$$u\in H^1(Q_T)\cap L^2\left(0,T;H_0^1(0,l)\cap H^2(0,l)\right), \tag{5}$$
$$\rho\in C^0(Q_T),\quad \rho_t,\rho_x\in L^{\infty,2}(Q_T),\quad \rho>0 \tag{6}$$

with the mass conservation

$$\int_0^l\rho(t,x)\,\mathrm{d}x=\int_0^l\rho^0(x)\,\mathrm{d}x=m,$$

and the energy equality

$$d_tE(t)+\mu\int_0^l(u_x)^2\,\mathrm{d}x=\int_0^l\rho g u\,\mathrm{d}x \tag{7}$$

denoting $E(t)=\int_0^l(1/2\rho u^2+P(\rho)-\rho F)\,\mathrm{d}x$ where $F(x)=If_\infty(x)=\int_0^x f_\infty(y)\,\mathrm{d}y$, $P(r)=r\int_1^r\frac{p(s)-p(1)}{s^2}\,\mathrm{d}s$, and assuming a natural structure for f, $f=f_\infty+g$ with $f_\infty\in W^{1,\infty}(0,l)$, and $g\in L^{2,\infty}(Q_\infty)$ expected to tend to zero.

As a consequence of (7), the following three properties are well-known and easily established either in the Lagrangian mass coordinates or in the Eulerian ones:

(i) estimates for $\|\sqrt{\rho}u\|_{L^{\infty,2}}$, $\|P(\rho)\|_{L^{\infty,1}}$ and $\|u_x\|_{L^2(Q_\infty)}$,

(ii) uniform upper bound for the density,
(iii) convergence to zero as $t \to \infty$ for the total kinetic energy $\|\sqrt{\rho}u(t,\cdot)\|_{L^2(0,l)}$.

There are many results about the behaviour of solutions to Eqs. (1), (2) under different boundary conditions, we refer e.g. [7] and [5] and the references therein.

Remark 1. Another form of the energy equality (7) is

$$d_t \widetilde{E}(t) + \mu \int_0^l (u_x)^2 \,\mathrm{d}x = \int_0^l \rho g u \,\mathrm{d}x \tag{8}$$

where $\widetilde{E}(t) = \int_0^l (1/2\rho u^2 + \rho \Pi(\rho, \rho_\infty))\,\mathrm{d}x$, a nice equivalent formulation because of

$$\rho \Pi(\rho, \rho_\infty) = \rho \int_{\rho_\infty}^{\rho} \big(p(s) - p(\rho_\infty)\big)/s^2 \,\mathrm{d}s \geqslant k|\rho - \rho_\infty|^\beta$$

which holds with a suitable constant $k = k(\beta)$, $\beta \geqslant \max(2, \gamma)$.

The rest states are now $(u_\infty, \rho_\infty) = (0, \rho_\infty)$ and the related stationary model is

$$p(\rho_\infty)_x = \rho_\infty f_\infty \quad \text{in } (0,l), \tag{9}$$

$$\rho_\infty \geqslant 0 \quad \text{and} \quad \int_0^l \rho_\infty(x)\,\mathrm{d}x = m. \tag{10}$$

Eq. (9) can be rewritten in the form $(\int_1^{\rho_\infty} p'(r)/r \,\mathrm{d}r)_x = f_\infty$; various assumptions on f_∞ and p have been studied to solve (9), (10).

Let us recall the following preliminary theorem devoted to necessary and sufficient conditions for the solution ρ_∞: Denoting $C_p = \int_0^1 p(r)/r^2 \,\mathrm{d}r \leqslant +\infty$, $F_{\min} = \min(F(x)\colon\ 0 \leqslant x \leqslant l)$, $F_{\max} = \max(F(x)\colon\ 0 \leqslant x \leqslant l)$, and $\Psi(r) = p(r)/r + \int_0^r p(s)/s \,\mathrm{d}s$ for $r > 0$, $\Psi(0) = 0$ (one can observe that for $C_p < +\infty$ the function $\Psi(\cdot)$ is continuous and increasing on $\mathbb{R}^+$), we get:

Theorem 2. *Part* 1. *A positive solution ρ_∞ to the problem* (9), (10) *exists if and only if one has either the three inequalities $C_p < +\infty$, $F_{\max} - F_{\min} < \Psi(\infty)$ and $1/m \int_0^l \Psi^{-1}(F(x) - F_{\min})\,\mathrm{d}x < 1$ where $\Psi^{-1}(\cdot)$ is the inverse of $\Psi(\cdot)$ or $C_p = +\infty$.*

Part 2. *For $p(\cdot)$ of class C^1 on $(0,\infty)$ and $F(\cdot)$ locally Lipschitz continuous on $(0,l)$, if $C_p < +\infty$ and if the upper level sets $\{x \in (0,l)\colon\ If_\infty(x) > \kappa\}$ are connected in $(0,l)$ for any $\kappa \in \mathbb{R}$, then there is at most one solution $\rho_\infty \in L^\infty_{\mathrm{loc}}(0,l)$ satisfying* (9), (10). *In case of existence, it is explicit $\rho_\infty(x) = \Psi^{-1}(\max(If_\infty(x) - k_l, 0))$ for a certain constant k_l.*

We refer to [1] for Part 1, to [2,3] for Part 2.

Remark 3. One can observe that if $p(r) = ar^\gamma$ then the conditions of Theorem 2 for the function $p(\cdot)$ are automatically satisfied: either $C_p < +\infty$ for $\gamma > 1$ and $\Psi(r) = a\gamma/(\gamma - 1)r^{\gamma-1}$, $\Psi(\infty) = \infty$, or $C_p = +\infty$ for $0 < \gamma \leqslant 1$, which correspond to regular cases, i.e. $\rho_\infty > 0$.

2. New energy functionals and the main result

As one shall see, the introduction of a quasi-stationary density $\bar{\rho} = \bar{\rho}(t,x)$ defined by

$$p(\bar\rho)_x = \rho f_\infty \quad \text{in } (0,l), \tag{11}$$

$$\int_0^l p\big(\bar\rho(t,x)\big)\,\mathrm{d}x = \int_0^l p\big(\rho(t,x)\big)\,\mathrm{d}x \quad \text{for all } t>0, \tag{12}$$

is one key of our approach.

It may be checked by a direct computation that (11), (12) determine explicitly $p(\bar\rho)$ in the form

$$p\big(\bar\rho(t,x)\big) = c(t) - \int_x^l \rho(t,y) f_\infty(y)\,\mathrm{d}y, \tag{13}$$

$$\text{with } c(t) = 1/l \int_0^l p\big(\rho(t,x)\big)\,\mathrm{d}x + \frac{1}{l}\int_0^l \left(\int_x^l \rho(t,y) f_\infty(y)\,\mathrm{d}y\right)\mathrm{d}x. \tag{14}$$

In the same way as in [8], one can show that:

Theorem 4. *For $p(\cdot)$, f_∞ in $BV([0,l])$ and chosen such that there is a unique solution to* (9), (10)*, we have the stabilizing properties in $L^q(0,l)$-norm,* $1 \leqslant q < \infty$,

$$\big\|\rho(t) - \rho_\infty\big\|_q \to 0 \quad \textit{as } t\to\infty,$$
$$\big\|p\big(\rho(t,\cdot)\big) - p\big(\bar\rho(t,\cdot)\big)\big\|_q \to 0 \quad \textit{as } t\to\infty,$$
$$\sup_x \big|p\big(\bar\rho(t,x)\big) - p(\rho_\infty)(x)\big| \to 0 \quad \textit{as } t\to\infty.$$

In view of the definition of new energy functionals, the question now is how do $\|p(\rho)-p(\bar\rho)\|_2^2$ and $\|\rho-\rho_\infty\|_\alpha^\alpha$ (with $\alpha=2$ or β) play same roles under the operator $\mathrm{d}/\mathrm{d}t$. To this end, we can describe the time–space evolution of $p(\rho)-p(\bar\rho)$ using an appropriate renormalized equation of continuity for $p(\rho)$, see hereafter (two words about the proof of Theorem 8).

Coming back to Eq. (2), we have

$$(\rho u)_t + \big(\rho u^2\big)_x - \mu u_{xx} + p(\rho)_x - p(\bar\rho)_x = \rho g. \tag{15}$$

This equation is interesting since, taking the product by u and integrating over $(0,l)$, we can compare with the energy equality (8) and conjecture

$$d_t \int_0^l \rho\Pi(\rho,\rho_\infty)\,\mathrm{d}x + \int_0^l \big(p(\rho)-p(\bar\rho)\big)u_x\,\mathrm{d}x = 0,$$

a result which is true (see [6]) under the assumptions

(V) the set $\{x\in(0,l)\colon\ \rho_\infty(x)=0\}$ is of measure zero, and (H) $\limsup_{r\to0^+}\int_0^r \frac{p'(s)}{s}\,\mathrm{d}s < \infty$.

Example. In this example we show simple solutions of (9), (10) which satisfy or do not satisfy the condition (V) although they are limits of the evolution solutions to (1)–(4) with a strictly positive initial density.

Let $l=1$, $p(\rho)=a\rho^\gamma$ with $a>0, \gamma>1$ constants $f_\infty(x)=f_0=\text{const}>0$ for $x\in[0,1/2]$, $f_\infty(x)=-f_0$ for $x\in(1/2,1]$. Put also $F(x)=f_0x$ in $[0,1/2]$ and $F(x)=f_0(1-x)$ in $(1/2,1]$. According to Theorem 2 the unique solution of the problem (9), (10) is given by $\rho_\infty(x)=\Psi^{-1}(\max(F(x)-k,0))$ with a suitable constant k. Consider the case when $k\in[0,f_0/2]$. Then elementary calculations yield $\rho_\infty(x)=0$ for $x\in[0,k/f_0]\cup[1-\frac{k}{f_0},1]$, and $\rho_\infty(x)=(\frac{\gamma-1}{a\gamma})^{\frac{1}{\gamma-1}}(f_0x-k)^{\frac{1}{\gamma-1}}$, if $x\in(k/f_0,1/2]$ and $\rho_\infty(x)=(\frac{\gamma-1}{a\gamma})^{\frac{1}{\gamma-1}}[f_0(1-x)-k]^{\frac{1}{\gamma-1}}$, if $x\in[1/2,1-\frac{k}{f_0})$. The condition (10) is satisfied by the constant $k=f_0/2-a^{1/\gamma}\frac{\gamma}{\gamma-1}(mf_0/2)^{1/\gamma}$. The condition $k\geqslant0$ yields $f_0\geqslant a(\frac{\gamma}{\gamma-1})^\gamma m^{\gamma-1}$. We see that if $k>0$, the assumption (V) is *not* satisfied while for $k=0$ it *is* satisfied since then there are only two points, $x=0,1$, where $\rho_\infty=0$. The latter case leads to the compatibility condition among the total mass, the external force, the adiabatic exponent γ and the physical constant a, namely, $f_0=a(\frac{\gamma}{\gamma-1})^\gamma m^{\gamma-1}$.

Definition 5. Define for strictly positive parameters ε, δ, η, the following energy–like functionals

$$E_{\varepsilon,\delta}(t)=\frac{1}{2}\int_0^l \big[\rho u^2+\delta\big(p(\rho)-p(\bar\rho)\big)^2-2\varepsilon\rho u I\big(p(\rho)-p(\bar\rho)\big)\big]\,\mathrm{d}x, \tag{16}$$

$$\widetilde{E}_{\varepsilon,\delta,\eta}(t)=\frac{1}{2}\int_0^l \big[\eta\big(\rho u^2+2\rho\Pi(\rho,\rho_\infty)\big)+\delta\big(p(\rho)-p(\bar\rho)\big)^2-2\varepsilon\rho u I\big(p(\rho)-p(\bar\rho)\big)\big]\,\mathrm{d}x. \tag{17}$$

It is not difficult to see

Lemma 6. $E_{\varepsilon,\delta}(t)\geqslant\min((1-2\varepsilon m\beta),(\delta-2\varepsilon ml/\beta))E_{0,1}(t)$. *For positivity, given m and l, one get some constraints in the choice of β and ε, δ.*

Then we claim our main result:

Theorem 7. *For all $t_0\geqslant 0$, for some $\alpha>0$ and $c>0$, one has the decay rate estimate*

$$\begin{aligned}&\|\sqrt{\rho}u\|_2^2+\|\rho-\rho_\infty\|_\beta^\beta+\big\|p(\rho)-p(\bar\rho)\big\|_2^2\\ &\quad\leqslant c\left\{\mathrm{e}^{-\alpha(t-t_0)}\left(1+\int_{t_0}^t \mathrm{e}^{\alpha s}\big\|g(s,\cdot)\big\|_2^2\,\mathrm{d}s\right)+\int_t^\infty\big\|g(s,\cdot)\big\|_2^2\,\mathrm{d}s\right\}\quad \text{for all } t>t_0\end{aligned}$$

with the assumptions above, $\beta\geqslant\max(\gamma,2)$, $\bar\rho$ defined by (11), (12)*, the hypothesis* (V) *and* (H)*, the uniqueness conditions for ρ_∞ as in Theorem* 2*, $f=f_\infty+g$, the initial data and all properties for $p(\cdot)$ as precised in the introduction.*

Therefore, if $g=0$ or if one controls the $L^2(Q)$-norm of $\mathrm{e}^{bt}g(t,x)$ for some $b\in(0,\alpha)$, the decay rate is exponential.

3. Main steps of the proof

We intend to perform a Lyapunov analysis, yielding for the ε-δ-η-family of functionals both appropriate choices of the parameters, especially ε_*, δ_*, η_*, and ordinary differential inequalities in the successive forms

$$\begin{aligned}&d_t\widetilde{E}_{\varepsilon,\delta,\eta}(t)+\widetilde{W}(t)\leqslant\widetilde{K}\|g\|\big|E_{\varepsilon,\delta}(t)\big|\quad\text{with }\widetilde{K}=\widetilde{K}\big(l,m,\varepsilon,\delta,\eta,\mu,E(0),\|f_\infty\|,\ldots\big),\\ &d_tE_{\varepsilon_*,\delta_*}(t)+W(t)\leqslant K\|g\|^2\quad\text{with }\ W(t)\geqslant\alpha E_{\varepsilon_*,\delta_*}(t).\end{aligned} \tag{18}$$

The construction procedure for a Lyapunov functional was known in the case of a density solution which are globally away from zero [8]. A careful nontrivial modification of the procedure leads to our result, so convenient now even if one misses a uniform lower bound for the density by a strictly positive constant. The elaboration of an adapted functional $\widetilde{E}_{\varepsilon_*,\delta_*,\eta_*}(\cdot)$ relates to a series of technical lemmas and to concrete refined estimates. A major point is a good description of all constants and an examination of their compatibility.

Let us stress some facts, collected together in

Theorem 8.

$$\begin{aligned}&d_tE_{\varepsilon,\delta}(t)+\eta\mu\|u_x\|_2^2+\eta\int_0^l(p(\bar\rho))-p(\rho))u_x\,\mathrm{d}x\\ &\quad+\varepsilon\int_0^l\rho uI(p(\rho)_t-p(\bar\rho))_t)\,\mathrm{d}x+\varepsilon\int_0^l(\rho u^2-\mu u_x)(p(\rho)-p(\bar\rho))\,\mathrm{d}x+\varepsilon\big\|p(\rho)-p(\bar\rho)\big\|_2^2\end{aligned}$$

$$+\,\delta/2\int_0^l \big(p(\bar\rho)^2-p(\rho)^2\big)u_x\,\mathrm{d}x+\delta\int_0^l \big(p(\rho)-p(\bar\rho)\big)\big(p(\rho)-\rho p'(\rho)\big)u_x\,\mathrm{d}x$$

$$+\,\delta\int_0^l \big(p(\rho)-p(\bar\rho)\big)\left(\int_x^l \rho u f'_\infty\,\mathrm{d}y\right)\mathrm{d}x=\eta\int_0^l \rho g u\,\mathrm{d}x+\varepsilon\int_0^l \rho g I\big(p(\bar\rho)\big)-p(\rho)\big)\,\mathrm{d}x.$$

Two words about the proof : We integrate over $(0,l)$ the product of Eq. (15) by $-\varepsilon I(p(\rho)-p(\bar\rho))$, and we integrate the product of

$$p(\rho)_t-p(\bar\rho))_t=-p(\rho)_x-p(\rho)u_x+\big(p(\rho)-\rho p'(\rho)\big)u_x-c'(t)-\int_x^l (\rho u)_x f_\infty\,\mathrm{d}y$$

by $\delta(p(\rho)-p(\bar\rho))$.

Next, all these new terms, on the left-hand side of equation given by Theorem 8, must be controlled in accordance with $\|u_x\|_2$ and $\|p(\rho)-p(\bar\rho)\|_2$.

Finally, there exist $\varepsilon_*>0$, $\delta_*>0$, $\eta_*>0$, and $\alpha_*>0$ for which we arrive at the differential inequality (18). Precisely $\delta_*=\varepsilon_*^{3/4}$, $\varepsilon_*\sim\eta_*^{4/5}$, and α_* is inversely proportional to $(\delta_*+ml(\eta_*+\varepsilon_*))$; these arguments are constructive. Moreover $\varepsilon_*,\delta_*,\eta_*,\alpha_*$ and K are locally bounded functions of $l,m,\mu,E(0),\|f_\infty\|_{W^{1,\infty}(0,l)}$, large data are available.

Thus $E_{\varepsilon_*,\delta_*}$ is a Lyapunov functional, and the decay rate estimate immediately follows. The 'a priori more natural' generalized energy functional $\widetilde{E}_{\varepsilon,\delta,1}(\cdot)$ seems not to enable us to obtain inequality (18).

The last step relies on the identity $d_t\int_0^l \rho\Pi(\rho,\rho_\infty)\,\mathrm{d}x+\int_0^l (p(\rho)-p(\bar\rho))u_x\,\mathrm{d}x=0$ under the assumption (V) $\{x\in(0,l)\colon\ \rho_\infty(x)=0\}$ of measure zero. So be the proof of Theorem 7 complete. The details of all proofs can be found in the paper [6].

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References

[1] H. Beirao da Veiga, An L^p-theory for the n-dimensional, stationary, compressible Navier–Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solutions, Commun. Math. Phys. 109 (1987) 229–248.

[2] R. Erban, On the static–limit solutions to the Navier–Stokes equations of compressible flow, J. Math Fluid Mech. 3 (2001) 393–408.

[3] E. Feireisl, H. Petzeltová, On the zero-velocity-limit of solutions to the Navier–Stokes equations of compressible flow, Manuscr. Math. 97 (1998) 109–116.

[4] A. Novotný, I. Straškraba, Stabilization of solutions to compressible Navier–Stokes equations, J. Math. Kyoto Univ. 40 (2) (2000) 217–245.

[5] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow, Oxford University Press, Oxford, 2003.

[6] P. Penel and I. Straškraba, Construction of a Lyapunov functional for 1D-viscous compressible barotropic fluid equations admitting vacua, Preprint of the Mathematical Institute of the Czech Academy of Sciences, Prague, 2006, 14 pp.

[7] I. Straškraba, Asymptotic development of vacuums for 1-d Navier–Stokes equations of compressible flow, Nonlinear World 3 (1996) 519–535.

[8] I. Straškraba, A.A. Zlotnik, On a decay rate for 1D-viscous compressible barotropic fluid equations, J. Evolution Equations 2 (2002) 69–96.