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Algebraic Geometry

M-regularity of the Fano surface

Andreas Höring

IRMA, université Louis Pasteur, 7, rue René Descartes, 67084 Strasbourg, France

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Abstract

In this Note we show that the Fano surface in the intermediate Jacobian of a smooth cubic threefold is M-regular in the sense of Pareschi and Popa. ***To cite this article: A. Höring, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

M*-régularité de la surface de Fano.** Dans cette Note, nous montrons que la surface de Fano dans la jacobienne intermédiaire d'une hypersurface cubique lisse de dimension trois est M-régulière au sens de Pareschi et Popa. ***Pour citer cet article : A. Höring, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

Version française abrégée

Soit $X^3 \subset \mathbb{P}^4$ une hypersurface cubique lisse. Sa jacobienne intermédiaire

$$J(X) := H^{2,1}(X, \mathbb{C})^*/H_3(X, \mathbb{Z})$$

est une variété abélienne principalement polarisée $(J(X), \Theta)$ de dimension cinq qui n'est pas la jacobienne d'une courbe [4, Thm. 0.12]. Soit F le schéma de Fano qui paramètre les droites contenues dans X. Alors F est une surface lisse et l'application d'Abel–Jacobi $F \to J(X)$ est un plongement qui induit un isomorphisme $\mathrm{Alb}(F) \simeq J(X)$ [4, Thm. 0.6,0.9]. De plus la classe de cohomologie de F dans $J(X)$ est minimale, c'est-à-dire

$$[F] = \frac{\Theta^3}{3!}.$$

On connaît une seule autre famille d'exemples de variétés abéliennes principalement polarisées (A, Θ) de dimension n telles que pour $1 \leqslant d \leqslant n-2$, la classe de cohomologie minimale $\Theta^{n-d}/(n-d)!$ peut être représentée par un cycle effectif de dimension d : ce sont les jacobiennes de courbes $J(C)$. Pour ces dernières, les sous-variétés $W_d(C) \subset J(C)$ sont de classe minimale. O. Debarre a montré que sur une jacobienne de courbe, ce sont les seules sous-variétés de classe minimale [5, Thm. 5.1]. Un théorème de Z. Ran [10, Thm. 5] montre que parmi les variétés abéliennes

E-mail address: andreas.hoering@ujf-grenoble.fr.

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principalement polarisées de dimension quatre, seules les (produits de) jacobiennes de courbes admettent une sous-variété de classe minimale. En dimension supérieure, peu de choses sont connues sur les sous-variétés de classe minimale.

Dans l'article [8], G. Pareschi et M. Popa introduisent une nouvelle approche pour caractériser ces sous-variétés : ils proposent de considérer les propriétés cohomologiques du faisceau de structure de la sous-variété tordu avec la polarisation. Plus précisement, on a la conjecture suivante :

Conjecture. [5,8] *Soit (A, Θ) une variété abélienne principalement polarisée irréductible de dimension n et soit Y une sous-variété non-dégénérée (cf.* [10, p. 464]*) de A de dimension $d \leqslant n-2$. Les propriétés suivantes sont équivalentes.*

(1) *La variété Y est de classe de cohomologie minimale, i.e. $[Y] = \Theta^{n-d}/(n-d)!$.*
(2) *Le faisceau de structure tordu $\mathcal{O}_Y(\Theta)$ est M-régulier (cf. Définition* 1.3 *ci-dessous), et $h^0(Y, \mathcal{O}_Y(\Theta) \otimes P_\xi) = 1$ pour $P_\xi \in \operatorname{Pic}^0(A)$ général.*
(3) *Soit (A, Θ) est la jacobienne d'une courbe lisse projective de dimension n et Y est un translaté de $W_d(C)$ ou $-W_d(C)$; soit $n = 5$, $d = 2$ et (A, Θ) est la jacobienne intermédiaire d'une hypersurface cubique lisse de dimension trois et Y est un translaté de F ou $-F$.*

L'implication (2) ⇒ (1) est l'objet de [8, Thm. B]. L'implication (3) ⇒ (2) a été démontrée pour les jacobiennes de courbes dans [7, Prop. 4.4]. Nous complétons la preuve de cette implication en traitant le cas de la jacobienne intermédiaire :

Théorème. *Soit $X^3 \subset \mathbb{P}^4$ une hypersurface cubique lisse et soit $(J(X), \Theta)$ sa jacobienne intermédiaire. Soit $F \subset J(X)$ la surface de Fano de X, plongée via une application d'Abel–Jacobi dans $J(X)$. Alors $\mathcal{O}_F(\Theta)$ est M-régulier et $h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) = 1$ pour $P_\xi \in \operatorname{Pic}^0 J(X)$ général.*

Puisque les propriétés considérées sont invariantes par isomorphisme, le théorème implique le même énoncé pour $-F$.

La preuve du théorème est basée sur la construction due à A. Beauville [3] de la surface de Fano comme une sous-variété spéciale de $J(X)$: on considère $J(X)$ comme la variété de Prym associée à un revêtement étale $\pi : \tilde{C} \to C$. La surface de Fano peut alors être décrite comme le sous-ensemble de $J(X)$ qui paramètre les diviseurs D sur $\tilde{C}$ tels que $\pi_* D = H$, où H est une section hyperplane de C (cf. Section 2 pour les détails). Puisque la courbe de Prym $\tilde{C}$ est aussi un diviseur de F, le calcul de la cohomologie de $\mathcal{O}_F(\Theta) \otimes P_\xi$ se ramène à une discussion des sections globales de $\mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi$ et $\mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi$ (étapes 1 et 3 de la preuve dans Section 3). Le résultat de ce calcul est assez frappant : les lieux où la cohomologie de $\mathcal{O}_F(\Theta) \otimes P_\xi$ ne s'annule pas sont des translatés de F.

1. Introduction

Let $X^3 \subset \mathbb{P}^4$ be a smooth cubic threefold, then its intermediate Jacobian

$$J(X) := H^{2,1}(X, \mathbb{C})^* / H_3(X, \mathbb{Z})$$

is a principally polarised Abelian variety $(J(X), \Theta)$ of dimension five that is not a Jacobian of a curve [4, Thm. 0.12]. The Fano scheme F parameterising lines contained in X is a smooth surface, and the Abel–Jacobi map $F \to J(X)$ is an embedding that induces an isomorphism $\operatorname{Alb}(F) \simeq J(X)$ [4, Thm. 0.6,0.9]. Furthermore the cohomology class of $F \subset J(X)$ is minimal, that is

$$[F] = \frac{\Theta^3}{3!}.$$

There is only one other known family of examples of principally polarised Abelian varieties (A, Θ) of dimension n such that for $1 \leqslant d \leqslant n-2$, a minimal cohomology class $\Theta^{n-d}/(n-d)!$ can be represented by an effective cycle of dimension d: the Jacobians of curves $J(C)$ where the subvarieties $W_d(C) \subset J(C)$ have minimal cohomology class. O. Debarre has shown that on a Jacobian these are the only subvarieties having minimal class [5, Thm. 5.1], furthermore by a theorem of Z. Ran [10, Thm. 5], the only principally polarised Abelian fourfolds with a subvariety of minimal class are (products of) Jacobians of curves. In higher dimension few things are known about subvarieties having minimal class.

In [8], G. Pareschi and M. Popa introduce a new approach to the characterisation of these subvarieties: they consider the (probably more tractable) cohomological properties of the twisted structure sheaf of the subvariety. More precisely we have the following conjecture:

Conjecture 1.1. [5,8] *Let (A, Θ) be an irreducible principally polarised Abelian varieties of dimension n, and let Y be a nondegenerate subvariety* (*cf.* [10, p.464]) *of A of dimension $d \leqslant n-2$. The following statements are equivalent.*

(1) *The variety Y has minimal cohomology class, i.e. $[Y] = \Theta^{n-d}/(n-d)!$.*
(2) *The twisted structure sheaf $\mathcal{O}_Y(\Theta)$ is M-regular* (*cf. Definition* 1.3 *below*), *and $h^0(Y, \mathcal{O}_Y(\Theta) \otimes P_\xi) = 1$ for $P_\xi \in \mathrm{Pic}^0(A)$ general.*
(3) *Either (A, Θ) is the Jacobian of a curve of genus n and Y is a translate of $W_d(C)$ or $-W_d(C)$, or $n = 5$, $d = 2$ and (A, Θ) is the intermediate Jacobian of a smooth cubic threefold and Y is a translate of F or $-F$.*

The implication (2) ⇒ (1) is the object of [8, Thm. B]. The implication (3) ⇒ (2) has been shown for Jacobians of curves in [7, Prop. 4.4]. We complete the proof of this implication by treating the case of the intermediate Jacobian:

Theorem 1.2. *Let $X^3 \subset \mathbb{P}^4$ be a smooth cubic threefold, and let $(J(X), \Theta)$ be its intermediate Jacobian. Let $F \subset J(X)$ be an Abel–Jacobi embedded copy of the Fano variety of lines in X. Then $\mathcal{O}_F(\Theta)$ is M-regular and $h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) = 1$ for $P_\xi \in \mathrm{Pic}^0 J(X)$ general.*

Since the properties considered are invariant under isomorphisms, the theorem implies the same statement for $-F$.

1.1. Notation and basic facts

We work over an algebraically closed field of characteristic different from 2 (cf. [1, chapitre 0] for the appropriate definitions in positive characteristic). We will denote by $D \equiv D'$ the linear equivalence of divisors, and by $D \equiv_{\mathrm{num}} D'$ the numerical equivalence.

For (A, Θ) a principally polarised Abelian variety (ppav), we identify A with $\hat{A} = \mathrm{Pic}^0(A)$ via the morphism induced by Θ. If $\xi \in A$ is a point, we denote by P_ξ the corresponding point in $\hat{A} = \mathrm{Pic}^0(A)$ which we consider as a numerically trivial line bundle on A.

Definition 1.3. [9,10 Lemma 3.8] Let (A, Θ) be a ppav of dimension n, and let $\mathcal{F}$ be a coherent sheaf on A. For all $n \geqslant i > 0$, we denote by

$$V^i_{\mathcal{F}} := \{\xi \in A \mid h^i(A, \mathcal{F} \otimes P_\xi) > 0\}$$

the i-th cohomological support locus of $\mathcal{F}$. We say that $\mathcal{F}$ is M-regular if

$$\operatorname{codim} V^i_{\mathcal{F}} > i$$

for all $i \in \{1, \ldots, n\}$.

If $l \subset X$ is a line, we will denote by $[l]$ the corresponding point of the Fano surface F and by $D_l \subset F$ the incidence curve of l, that is, D_l parametrises lines in X that meet l. Furthermore we have by [4, §10], [11, §6] and Riemann–Roch that

$$\mathcal{O}_F(\Theta) \equiv_{\mathrm{num}} 2D_l, \tag{1}$$
$$K_F \equiv_{\mathrm{num}} 3D_l, \tag{2}$$
$$D_l \cdot D_l = 5, \tag{3}$$
$$\chi(F, \mathcal{O}_F(\Theta)) = 1. \tag{4}$$

2. Prym construction of the Fano surface

We recall the construction of the Fano surface as a special subvariety of a Prym variety [3,2]: let $\tilde{C} := D_{l_0} \subset F$ be the incidence curve of a general line $l_0 \subset X$. Let X' be the blow-up of X in l_0. Then the projection from l_0 induces a conic bundle structure $X' \to \mathbb{P}^2$ with branch locus $C \subset \mathbb{P}^2$ a smooth quintic. This conic bundle induces a natural connected étale covering of degree two $\pi : \tilde{C} \to C$ (cf. [1, Ch. I] for details), and we denote by $\sigma : \tilde{C} \to \tilde{C}$ the involution induced by π.

The kernel of the norm morphism $\mathrm{Nm} : J\tilde{C} \to JC$ has two connected components which we will denote by P and P_1. The zero component P is called the Prym variety associated to π, and it is isomorphic as a ppav to $J(X)$ [1, Thm. 2.1].

Let $H \subset C$ be an effective divisor given by a hyperplane section in $\mathbb{P}^2$. Then H has degree five and $h^0(C, \mathcal{O}_C(H)) = 3$, so the complete linear system g_5^2 corresponds to a $\mathbb{P}^2 \subset C^{(5)}$. We choose a divisor $\tilde{H} \in \tilde{C}^{(5)}$ such that $\pi^{(5)}([\tilde{H}]) = [H]$, where $\pi^{(5)} : \tilde{C}^{(5)} \to C^{(5)}$ is the morphism induced by π on the symmetric products. Let $\phi_H : C^{(5)} \to JC$ and $\phi_{\tilde{H}} : \tilde{C}^{(5)} \to J\tilde{C}$ be the Abel–Jacobi maps given by H and $\tilde{H}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{C}^{(5)} & \xrightarrow{\phi_{\tilde{H}}} & J\tilde{C} \\ \downarrow{\scriptstyle \pi^{(5)}} & & \downarrow{\scriptstyle \mathrm{Nm}} \\ C^{(5)} & \xrightarrow[\phi_H]{} & JC \end{array}$$

The fibre of $\phi_{\tilde{H}}(\tilde{C}^{(5)}) \to \phi_H(C^{(5)})$ over the point 0 (and thus the intersection of $\phi_{\tilde{H}}(\tilde{C}^{(5)})$ with $\ker \mathrm{Nm}$) has two connected components $F_0 \subset P$ and $F_1 \subset P_1$. If we identify P and P_1 via $\tilde{H} - \sigma(\tilde{H})$, we obtain an identification $F_1 = -F_0$ [3, p. 360]. The (non-canonical) isomorphism of ppavs $P \simeq J(X)$ transforms F_0 into a translate of the Fano surface F [3, Thm. 4].

From now on we will identify P (resp. F_0) and $J(X)$ (resp. some Abel–Jacobi embedded copy of the Fano surface F).

We will now prove two technical lemmata on certain linear systems on $\tilde{C}$. The first is merely a reformulation of [2, §2, ii)].

Lemma 2.1. *The line bundle $\mathcal{O}_{\tilde{C}}(\tilde{C})$ is a base-point free pencil of degree five such that any divisor $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C})|$ satisfies $\pi_* D \equiv H$.*

Proof. We define a morphism $\mu : \tilde{C} = D_{l_0} \to l_0 \simeq \mathbb{P}^1$ by sending $[l] \in \tilde{C}$ to $l \cap l_0$. Since l_0 is general and through a general point of l_0 there are five lines distinct from l_0, the morphism μ has degree 5. If $[l] \in F$, then $D_l \cdot D_{l_0} = 5$ by formula (3), so for $[l] \neq [l_0]$ the divisor $D_{l_0} \cap D_l \in |\mathcal{O}_{\tilde{C}}(D_l)|$ is effective. Furthermore $\pi_* D_l \equiv H$, since $\pi_* D_l$ is the intersection of $C \subset \mathbb{P}^2$ with the image of l under the projection $X' \to \mathbb{P}^2$. By specialisation the linear system $|\mathcal{O}_{\tilde{C}}(\tilde{C})|$ is not empty and a general divisor D in it corresponds to the five lines distinct from l_0 passing through a general point of l_0. Hence $\mathcal{O}_{\tilde{C}}(\tilde{C}) \simeq \mu^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\pi_* D \equiv H$. □

Lemma 2.2. *The sets*

$$V_0' := \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi) > 0\},$$
$$V_1' := \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) > 1\}$$

are contained in translates of $F \cup (-F)$.

Proof. (1) Let $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi|$ be an effective divisor. Then $\pi_* \tilde{C} \equiv \pi_* D \equiv H$. It follows that $D \in (\phi_{\tilde{H}}(\tilde{C}^{(5)}) \cap \ker \mathrm{Nm})$, so D is in F or $-F$.

(2) We follow the argument in [2, §3]. By [2, §2, iv)] we have $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C} + \sigma(\tilde{C}))) = 4$, so $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}))$ is odd. It follows from the deformation invariance of the parity [6, p. 186f] that

$$V_1' = \{\xi \in P \mid h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) \geqslant 3\}.$$

Fix $\xi \in P$ such that $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) \geqslant 3$ and $D \in |\mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi|$. Let s and t be two sections of $\mathcal{O}_{\tilde{C}}(\tilde{C})$ such that the associated divisors have disjoint supports, then we have an exact sequence

$$0 \to \mathcal{O}_{\tilde{C}}(D - \tilde{C}) \xrightarrow{(t,-s)} \mathcal{O}_{\tilde{C}}(D)^{\oplus 2} \xrightarrow{(s,t)} \mathcal{O}_{\tilde{C}}(D + \tilde{C}) \to 0.$$

This implies

$$h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \tilde{C})\big) + h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}(D + \tilde{C})\big) \geqslant 2h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}(D)\big) = 6,$$

furthermore by Riemann–Roch $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D + \tilde{C})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - D - \tilde{C})) + 5$. Now $K_{\tilde{C}} - D \equiv \sigma(D)$ and $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\sigma(D) - \tilde{C})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \sigma(\tilde{C})))$ imply

$$h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - \tilde{C})\big) + h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}\big(D - \sigma(\tilde{C})\big)\big) \geqslant 1.$$

Hence $D \equiv \tilde{C} + D'$ or $D \equiv \sigma(\tilde{C}) + D'$ where D' is an effective divisor such that $\pi_* D' \equiv H$. We see as in the first part of the proof that the effective divisors D' such that $\pi_* D' \equiv H$ are parametrised by a set that is contained in a translate of $F \cup (-F)$. □

3. Proof of Theorem 1.2

Since $\mathcal{O}_F(\Theta) \equiv_{\text{num}} \mathcal{O}_F(2\tilde{C})$ by formula (1), it is equivalent to verify the stated properties for the sheaf $\mathcal{O}_F(2\tilde{C})$.

Step 1. The second cohomological support locus is contained in a translate of $F \cup (-F)$. By formula (2), we have $K_F \equiv \mathcal{O}_F(3\tilde{C}) \otimes P_{\xi_0}$ for some $\xi_0 \in P$. Hence by Serre duality $h^2(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0})$, so it is equivalent to consider the non-vanishing locus

$$V_0 := \big\{\xi \in P \mid h^0\big(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi\big) > 0\big\}.$$

Note that if $l \in F$ is a line on X, the corresponding incidence curve $D_l \subset F$ is an effective divisor numerically equivalent to $\tilde{C}$, so it is clear that $\pm F$ is (up to translation) a subset of V_0. Consider now the exact sequence

$$0 \to \mathcal{O}_F \otimes P_\xi \to \mathcal{O}_F(\tilde{C}) \otimes P_\xi \to \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi \to 0.$$

Clearly $h^0(F, \mathcal{O}_F \otimes P_\xi) = 0$ for $\xi \neq 0$, so $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi) \leqslant h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\tilde{C}) \otimes P_\xi)$ for $\xi \neq 0$. Since a divisor $D \in |\mathcal{O}_{\tilde{C}}(\tilde{C})|$ satisfies $\pi_* D \equiv H$, we conclude with Lemma 2.2.

Step 2. The first cohomological support locus is contained in a union of translates of $F \cup (-F)$. Since $\chi(F, \mathcal{O}_F(2\tilde{C})) = \chi(F, \mathcal{O}_F(\Theta)) = 1$ (formula (4)), we have

$$h^1\big(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi\big) = h^0\big(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi\big) + h^0\big(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0}\big) - 1.$$

Since

$$h^0\big(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi\big) = h^0\big(F, \mathcal{O}_F(\Theta) \otimes P_\xi\big) \geqslant 1$$

for all $\xi \in P$, the first cohomological support locus is contained in the locus where $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi^* \otimes P_{\xi_0}) > 0$ or $h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) > 1$. By step 1 the statement follows if we show the following claim: the set

$$V_1 := \big\{\xi \in P \mid h^0\big(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi\big) > 1\big\}$$

is contained in a union of translates of $F \cup (-F)$.

Step 3. Proof of the claim and conclusion. Consider the exact sequence

$$0 \to \mathcal{O}_F(\tilde{C}) \otimes P_\xi \to \mathcal{O}_F(2\tilde{C}) \otimes P_\xi \to \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi \to 0.$$

By the first step we know that $h^0(F, \mathcal{O}_F(\tilde{C}) \otimes P_\xi) = 0$ for ξ in the complement of a translate of $F \cup (-F)$, so

$$h^0\big(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi\big) \leqslant h^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi\big)$$

for ξ in the complement of a translate of $F \cup (-F)$. The claim is then immediate from Lemma 2.2. By the same lemma $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2\tilde{C}) \otimes P_\xi) = 1$ for $\xi \in P$ general, so $h^0(F, \mathcal{O}_F(2\tilde{C}) \otimes P_\xi) = h^0(F, \mathcal{O}_F(\Theta) \otimes P_\xi) = 1$ for $\xi \in P$ general. □

Remark. It is possible to strengthen a posteriori the statements in the proof: since Theorem 1.2 holds, we can use the Fourier–Mukai techniques from [8] to see that the cohomological support loci are supported exactly on the *theta-dual* of F (ibid, Definition 4.2), which in our case is just F.

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