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Probability Theory

A characterization of the set-indexed fractional Brownian motion by increasing paths

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Abstract

We prove that a set-indexed process is a set-indexed fractional Brownian motion if and only if its projections on all the increasing paths are one-parameter time changed fractional Brownian motions. As an application, we present an integral representation for such processes. ***To cite this article: E. Herbin, E. Merzbach, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une caractérisation par chemins croissants du mouvement brownien fractionnaire indexé par des ensembles. On montre qu'un processus stochastique est un mouvement brownien fractionnaire indexé par des ensembles si et seulement si ses projections sur tous les chemins croissants sont des mouvements browniens fractionnaires à paramètres réels changés de temps. On applique ce résultat à la définition d'une représentation intégrale pour de tels processus. ***Pour citer cet article : E. Herbin, E. Merzbach, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Dans [1], le *mouvement brownien fractionnaire indexé par des ensembles (sifBm)* est défini et ses propriétés de stationnarité et d'autosimilarité sont étudiées. D'autre part, on prouve que la projection d'un sifBm sur un chemin croissant est un mouvement brownien fractionnaire indexé par $\mathbf{R}_+$ changé de temps. L'objet de cette note est la réciproque de ce résultat.

En considérant une collection d'indices $\mathcal{A}$ de sous-ensembles compacts d'un espace métrique localement compact $\mathcal{T}$ muni d'une mesure de Radon m (voir [1]), le mouvement brownien fractionnaire indexé par $\mathcal{A}$ est défini comme le processus gaussien centré $\mathbf{B}^H = \{\mathbf{B}_U^H;\, U \in \mathcal{A}\}$ tel que

$$\forall U, V \in \mathcal{A}; \quad E\big[\mathbf{B}_U^H \mathbf{B}_V^H\big] = \frac{1}{2}\big[m(U)^{2H} + m(V)^{2H} - m(U \bigtriangleup V)^{2H}\big], \tag{1}$$

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où $0 < H \leqslant \frac{1}{2}$ (lorsque $H > \frac{1}{2}$, (1) ne définit pas un processus, quelle que soit la collection indexante non totalement ordonnée $\mathcal{A}$).

Définition 0.1. On appelle *flot élémentaire* toute fonction $f : [a,b] \subset \mathbf{R}_+ \to \mathcal{A}$ vérifiant

$$\forall s,t \in [a,b]; \quad s < t \Rightarrow f(s) \subseteq f(t),$$
$$\forall s \in [a,b); \quad f(s) = \bigcap_{v>s} f(v),$$
$$\forall s \in (a,b); \quad f(s) = \overline{\bigcup_{u<s} f(u)}.$$

Définition 0.2. Un processus indexé par des ensembles $X = \{X_U; U \in \mathcal{A}\}$ est dit *continu monotone extérieurement dans* L^2 si X_U est carré intégrable pour tout $U \in \mathcal{A}$ et pour toute suite décroissante $(U_n)_{n\in\mathbf{N}}$ d'ensembles dans $\mathcal{A}$,

$$E\big[|X_{U_n} - X_{\bigcap_m U_m}|^2\big] \to 0$$

quand $n \to \infty$.

Théorème 0.3. *Soit $X = \{X_U; U \in \mathcal{A}\}$ un processus continu monotone extérieurement dans L^2.*

Si la projection X^f de X sur tout flot élémentaire f, est, à un changement de temps près, un mouvement brownien fractionnaire indexé par $\mathbf{R}_+$ de paramètre $H \in (0, 1/2)$, alors X est un mouvement brownien fractionnaire indexé par $\mathcal{A}$.

Cette caractérisation fournit une bonne justification de la définition du sifBm et ouvre la porte à une grande variété d'applications. La représentation intégrale (13) constitue l'une d'entre elles.

1. Introduction

In [1], the set-indexed fractional Brownian motion (sifBm) is defined and its properties of stationarity and self-similarity are discussed. In particular, it is proved that the projection of a sifBm on an increasing path is a one-parameter time changed fractional motion. In this note, we prove the converse.

This characterization gives a good justification of the definition of the sifBm and opens the door to a variety of applications. Here we present one of them: an integral representation for the sifBm.

We follow [1] for the framework and notation. Our processes are indexed by an indexing collection $\mathcal{A}$ of compact subsets of a locally compact metric space $\mathcal{T}$ equipped with a Radon measure m.

The *set-indexed fractional Brownian motion (sifBm)* was defined as the centered Gaussian process $\mathbf{B}^H = \{\mathbf{B}^H_U; U \in \mathcal{A}\}$ such that

$$\forall U, V \in \mathcal{A}; \quad E\big[\mathbf{B}^H_U \mathbf{B}^H_V\big] = \frac{1}{2}\big[m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}\big], \tag{2}$$

where $0 < H \leqslant \frac{1}{2}$ (when $H > \frac{1}{2}$, (2) can not define a stochastic process, for any non-totally ordered indexing collection $\mathcal{A}$).

If $\mathcal{A}$ is provided with a structure of group on $\mathcal{T}$, properties of stationarity and self-similarity are studied in [1]. In the special case of $\mathcal{A} = \{[0,t]; t \in \mathbf{R}^N_+\} \cup \{\emptyset\}$, we get a multiparameter process called Multiparameter fractional Brownian motion (MpfBm), whose properties are studied in [2].

2. Projection of the sifBm on flows

The notion of flow is the key to reduce the proof of many theorems. It was extensively studied in [3] and [4].

Let $\mathcal{A}(u)$ denotes the class of finite unions from sets belonging to $\mathcal{A}$.

Definition 2.1. An *elementary flow* is defined to be a continuous increasing function $f : [a,b] \subset \mathbf{R}_+ \to \mathcal{A}$, i.e. such that

$$\forall s, t \in [a,b]; \quad s < t \Rightarrow f(s) \subseteq f(t),$$
$$\forall s \in [a,b); \quad f(s) = \bigcap_{v>s} f(v),$$
$$\forall s \in (a,b); \quad f(s) = \overline{\bigcup_{u<s} f(u)}.$$

A *simple flow* is a continuous function $f : [a,b] \to \mathcal{A}(u)$ such that there exists a finite sequence $(t_0, t_1, \ldots, t_n)$ with $a = t_0 < t_1 < \cdots < t_n = b$ and elementary flows $f_i : [t_{i-1}, t_i] \to \mathcal{A}$ $(i = 1, \ldots, n)$ such that

$$\forall s \in [t_{i-1}, t_i]; \quad f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).$$

The set of all simple (resp. elementary) flows is denoted $S(\mathcal{A})$ (resp. $S^e(\mathcal{A})$).

Proposition 2.2. ([1]) *Let $\mathbf{B}^H$ be a sifBm and f be an elementary flow. Then the process $(\mathbf{B}^H)^f = \{\mathbf{B}^H_{f(t)}, t \in [a,b]\}$ is a time changed fractional Brownian motion.*

The aim of this note is to prove the converse to Proposition 2.2. For this purpose, we will use the following lemma proved in [3].

Lemma 2.3. *The finite dimensional distributions of an additive $\mathcal{A}$-indexed process X determine and are determined by the finite dimensional distributions of the class $\{X^f, f \in S(\mathcal{A})\}$.*

3. Characterization of the sifBm

The converse to Proposition 2.2 in the case of L^2-monotone outer-continuous set-indexed processes, gives a characterization of the sifBm by its projection on elementary flows.

Recall the following definition (see [4])

Definition 3.1. A set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ is said L^2*-monotone outer-continuous* if X_U is square integrable for all $U \in \mathcal{A}$ and for any decreasing sequence $(U_n)_{n\in\mathbf{N}}$ of sets in $\mathcal{A}$,

$$E\big[|X_{U_n} - X_{\bigcap_m U_m}|^2\big] \to 0$$

as $n \to \infty$.

Theorem 3.2. *Let $X = \{X_U; U \in \mathcal{A}\}$ be a L^2-monotone outer-continuous set-indexed process.*

If the projection X^f of X on any elementary flow f, is a time-changed one-parameter fractional Brownian motion of parameter $H \in (0, 1/2)$, then X is a set-indexed fractional Brownian motion.

Proof. Let $f : [a,b] \to \mathcal{A}$ be an elementary flow. As the projected process X^f is a time-changed fBm of parameter H, we have

$$\forall s, t \in [a,b]; \quad E\big[X^f_t - X^f_s\big]^2 = \big|\theta_f(t) - \theta_f(s)\big|^{2H} \tag{3}$$

where $X^f_t = X_{f(t)}$ and θ_f is an increasing function.

The idea of the proof is the construction of a measure m such that for any $f \in S^e(\mathcal{A})$,

$$\forall t \in [a,b]; \quad \theta_f(t) = m\big[f(t)\big].$$

For all $U \in \mathcal{A}$, let us define

$$F^e_U = \big\{f \in S^e(\mathcal{A}) \colon \exists u_f \in [a,b]; U = f(u_f)\big\}.$$

As for all f and g in F^e_U, $\theta_f(u_f)^{2H} = \theta_g(u_g)^{2H} = E[X_U]^2$, one can define

$$\psi(U) = \theta_f(u_f) = \big(E[X_U]^2\big)^{1/(2H)}. \tag{4}$$

For all U and V in $\mathcal{A}$ with $U \subset V$, there exists an elementary flow f such that

$$\exists u_f, v_f \in [a,b]; \quad u_f \leqslant v_f; \quad U = f(u_f) \subset f(v_f) = V.$$

Then, as the time-change θ_f is increasing, ψ is non-decreasing in $\mathcal{A}$.

The definition of ψ on $\mathcal{A}$ can be extended to the collection $\mathcal{C}$ of sets of the form $C = U \setminus \bigcup_{1\leqslant i\leqslant n} U_i$ where $U, U_1, \dots, U_n \in \mathcal{A}$, by the inclusion–exclusion formula

$$\psi(C) = \psi(U) - \sum_{i=1}^{n} \psi(U \cap U_i) + \sum_{i<j} \psi\big(U \cap (U_i \cap U_j)\big) - \cdots + (-1)^n \psi\Bigg(U \cap \Bigg(\bigcap_{1\leqslant i\leqslant n} U_i\Bigg)\Bigg). \tag{5}$$

The definition (5) of ψ can be easily extended to the set $\mathcal{C}(u)$ of finite unions of elements of $\mathcal{C}$ in the same way. Then, for all $C_1, C_2 \in \mathcal{C}(u)$ such that $C = C_1 \cup C_2 \in \mathcal{C}$,

$$\psi(C_1 \cup C_2) = \psi(C_1) + \psi(C_2) - \psi(C_1 \cap C_2). \tag{6}$$

From the pre-measure ψ defined on $\mathcal{C}$, the function

$$m : E \subset \mathcal{T} \mapsto \inf_{\substack{C_i \in \mathcal{C}\\ E \subset \cup C_i}} \sum_{i=1}^{\infty} \psi(C_i) \tag{7}$$

defines an outer measure on $\mathcal{T}$ (see [5, pp. 9–26]). Let us show that m defines a Borel measure on the topological space $\mathcal{T}$.

Let $\mathcal{M}_m$ be the σ-field of m-measurable subsets of $\mathcal{T}$. It is known that m is a measure on $\mathcal{M}_m$ (see [5, Theorem 3]). By definition, any $U \in \mathcal{A}$ is m-measurable if

$$\forall A \subset U,\ \forall B \subset \mathcal{T} \setminus U; \quad m(A \cup B) = m(A) + m(B).$$

As the inequality $m(A \cup B) \leqslant m(A) + m(B)$ follows from definition of any outer-measure, it remains to show the converse inequality.

Consider any sequence $(C_i)_{i\in\mathbf{N}}$ in $\mathcal{C}$ such that $A \cup B \subset \bigcup_i C_i$. The sequence $(C_i)_{i\in\mathbf{N}}$ can be decomposed by the elements $C_i, i \in I$, such that $C_i \cap U = \emptyset$ and the $C_i, i \in J$, such that $C_i \subset U$ (if $C_i \cap U \neq \emptyset$ and $C_i \not\subset U$, cut $C_i = C_i' \cup C_i''$ where $C_i' \subset U$ and $C_i'' \cap U = \emptyset$).

From

$$A \cup B \subset \Bigg[\bigcup_{i\in I} C_i\Bigg] \cup \Bigg[\bigcup_{i\in J} C_i\Bigg],$$

we get

$$\sum_{i=1}^{\infty} \psi(C_i) = \underbrace{\sum_{i\in I} \psi(C_i)}_{\geqslant m(B)} + \underbrace{\sum_{i\in J} \psi(C_i)}_{\geqslant m(A)}$$

which leads to $m(A \cup B) \geqslant m(A) + m(B)$.

We have proved that $\mathcal{A} \subset \mathcal{M}_m$. By definition of $\mathcal{A}$, the smallest σ-field containing $\mathcal{A}$ is the Borel σ-field $\mathcal{B}$. Therefore, $\mathcal{B} \subset \mathcal{M}_m$ and m is a measure on $\mathcal{B}$.

The second part of the proof is to show that the measure m is an extension of ψ, i.e.

$$\forall U \in \mathcal{A}; \quad m(U) = \psi(U). \tag{8}$$

– For any $U \in \mathcal{A}$, by definition of $m(U)$,

$$m(U) = \inf_{\substack{C_i \in \mathcal{C}\\ U \subset \cup C_i}} \sum_{i=1}^{\infty} \psi(C_i) \leqslant \psi(U). \tag{9}$$

– To prove the converse inequality, consider $U \in \mathcal{A}$ and a sequence $(C_i)_{i\in\mathbf{N}}$ in $\mathcal{C}$ such that $U \subset \bigcup_i C_i$. For all $n \in \mathbf{N}^*$, we have

$$U \subset \bigcup_{1\leqslant i\leqslant n} C_i \cup \Bigg[U \Big\backslash \bigcup_{1\leqslant i\leqslant n} C_i\Bigg].$$

Then, (6) implies

$$\psi(U) \leqslant \sum_{i=1}^{\infty} \psi(C_i) + \psi\left(U \setminus \bigcup_{1\leqslant i\leqslant n} C_i\right). \tag{10}$$

Using L^2-monotone outer continuity of X and proposition 1.4.8 in [4], we have

$$\lim_{n\to\infty} \psi\left(U \setminus \bigcup_{1\leqslant i\leqslant n} C_i\right) = 0. \tag{11}$$

Thus, (10) and (11) imply that for all sequence $(C_i)_{i\in\mathbf{N}}$ in $\mathcal{C}$ such that $U \subset \bigcup_i C_i$,

$$\psi(U) \leqslant \sum_{i=1}^{\infty} \psi(C_i)$$

and then, by definition of $m(U)$

$$\psi(U) \leqslant m(U). \tag{12}$$

Equality (8) follows from (9) and (12).

From (4) and (8), the Borel measure m defined by (7) satisfies

$$\forall U \in \mathcal{A};\quad E[X_U]^2 = \psi(U)^{2H} = m(U)^{2H}.$$

Consider a set-indexed fractional Brownian motion Y, defined by

$$\forall U, V \in \mathcal{A};\quad E[Y_U Y_V] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}\right].$$

According to Proposition 6.4 in [1], projections of Y on any elementary flow $f\colon [a,b] \to \mathcal{A}$ is a time-change one-parameter fractional Brownian motion, i.e. such that

$$\begin{aligned}\forall s,t \in [a,b];\quad E\left[Y_t^f - Y_s^f\right]^2 &= \left|m\left[f(t)\right] - m\left[f(s)\right]\right|^{2H}\\ &= \left|\theta_f(t) - \theta_f(s)\right|^{2H},\end{aligned}$$

where the projection Y^f is defined by $Y_t^f = Y_{f(t)}$, for all t.

Then, the projections of the set-indexed processes X and Y on any elementary flow have the same distribution. By additivity, this fact holds also on any simple flow. Thus, lemma 2.3 implies X and Y have the same law. □

As a corollary, we get an integral representation.

Corollary 3.3 *(Integral Representation). Let $X = \{X_U; U \in \mathcal{A}\}$ be a L^2 outer-continuous set-indexed process. Then, X is a sifBm if and only if for any $U \in \mathcal{A}$, there exist $f \in F_U^e$ and a Brownian motion W_f such that*

$$X_U = \int_{\mathbf{R}} \left(\left|m(U) - u\right|^{H-1/2} - |u|^{H-1/2}\right) W_f(\mathrm{d}u) \tag{13}$$

where $H \in [0, 1/2)$.

Proof. The implication is obvious. Let us prove the converse.

Let $U \in \mathcal{A}$, $\forall f \in F_U^e$, $\exists \theta_f\colon \theta_f(t) = m(f(t)) = m(U)$. Then

$$X_U = \mathbf{B}_{f(t)}^H = \left(\mathbf{B}^H\right)_t^f = \int_{\mathbf{R}} \left(\left|\theta_f(t) - u\right|^{H-1/2} - |u|^{H-1/2}\right) W_f(\mathrm{d}u),$$

and the result follows. □

Remark 1.

- If $H = 1/2$, formula (13) does not hold, but if we decompose $\mathbf{R}$ into negative and positive parts, the formula can be also interpreted for $H = 1/2$.
- As W_f depends on the flow f, expression (13) can not provide an integral representation of the whole set-indexed process $\mathbf{B}^H$, but only of its projection on a flow.

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