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Number Theory

A transcendence criterion in positive characteristic and applications

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Abstract

In this Note we shall present a general transcendence criterion in positive characteristic, which unifies many famous transcendence criteria such as those of L.I. Wade, S.M. Spencer, Jr., B. de Mathan, L. Denis, Y. Hellegouarch, etc. As applications, we shall study the transcendence of two families of functions at nonzero algebraic arguments. ***To cite this article: J.-Y. Yao, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Un critère de transcendance en caractéristique positive et applications. Dans cette Note nous présenterons un critère de transcendance général en caractéristique positive, qui unifie de nombreux critères de transcendance célèbres comme ceux de L.I. Wade, S.M. Spencer, Jr., B. de Mathan, L. Denis, Y. Hellegouarch, etc. Comme applications, nous étudierons la transcendance de deux familles de fonctions aux arguments algébriques non nuls. ***Pour citer cet article : J.-Y. Yao, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Soit K un corps de caractéristique $p \geqslant 2$ et $q = p^w$ avec $w \geqslant 1$ un entier. Désignons par $K[T]$ l'anneau intègre des polynômes en T à coefficients dans K. Son corps de fractions est noté $K(T)$.

Pour tous les $P, Q \in K[T]$ avec $Q \neq 0$, nous définisons

$$|P/Q|_\infty := q^{\deg(P)-\deg(Q)}.$$

Enfin nous désignons par $K((T^{-1}))$ le complété topologique de $K(T)$ pour $|\cdot|_\infty$, et par $\mathbf{C}_\infty$ le complété topologique d'une clôture algébrique fixée de $K((T^{-1}))$.

Le résultat principal de ce travail est le suivant, qui unifie de nombreux célèbres critères de transcendance comme ceux de L.I. Wade [11], S.M. Spencer, Jr. [10], B. de Mathan [5], L. Denis [6], Y. Hellegouarch [7], etc. :

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Théorème 1. *Soit $\alpha \in K((T^{-1}))$ tel qu'il existe deux suites $(P_n)_{n\geqslant 0}$ et $(Q_n)_{n\geqslant 0}$ d'éléments non nuls dans $K[T]$ satisfaisant aux conditions suivantes* :

1) *Il existe une suite $(\delta_n)_{n\geqslant 0}$ de nombres réels strictement positifs telle que*
$$|\alpha - P_n/Q_n|_\infty \leqslant \delta_n$$
pour tous les entiers $n \geqslant 0$;
2) *Pour tous les $(r+1)$-tuples $(A_0,\dots,A_r)$ d'éléments dans $K[T]$ avec $A_r \neq 0$, il existe un ensemble infini S d'entiers $n \geqslant r$ tels que*
$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{\theta_j(n)}}{Q_{\theta_j(n)}}\right)^{q^j}$$
soit différent de zéro, et que pour tous les entiers j $(0 \leqslant j \leqslant r)$, on ait
$$\limsup_{\substack{n\in S\\ n\to+\infty}} \frac{|\beta_n|}{\delta_{\theta_j(n)}^{q^j}} = +\infty,$$
où chaque $\theta_j = (\theta_j(n))_{n\geqslant 0}$ $(0 \leqslant j \leqslant r)$ est une suite d'entiers positifs strictement croissants vers $+\infty$.

Alors α est transcendant sur $K(T)$.

Fixons maintenant $K = \mathbb{F}_q$ le corps fini à q éléments. Soit $u = (u(n))_{n\geqslant 0}$ une suite à valeurs dans $\mathbb{F}_q$. Posons
$$\kappa_u^{(k)} := \sum_{n=0}^{+\infty} u(n) t^{k^n}, \quad \text{et} \quad \varpi_u^{(k)} := \sum_{n=0}^{+\infty} u(n) t^{n^k}.$$
Il est à noter que si la suite u n'est pas ultimement nulle, alors les rayons de convergence de ces deux séries formelles $\kappa_u^{(k)}$ et $\varpi_u^{(k)}$ sont égaux à 1.

Comme applications du Théorème 1, nous avons les deux résultats suivants :

Théorème 2. *Soit $u = (u(n))_{n\geqslant 0}$ une suite dans $\mathbb{F}_q$ qui n'est pas ultimement nulle. Soit $\alpha \in \mathbf{C}_\infty$ algébrique sur $\mathbb{F}_q(T)$ tel que $0 < |\alpha|_\infty < 1$. Alors $\kappa_u^{(k)}(\alpha)$ est algébrique sur $\mathbb{F}_q(T)$ si et seulement si k est une puissance de p et la suite u est ultimement périodique.*

Théorème 3. *Soit $u = (u(n))_{n\geqslant 0}$ une suite dans $\mathbb{F}_q$ qui n'est pas ultimement nulle. Soit $\alpha \in \mathbf{C}_\infty$ algébrique sur $\mathbb{F}_q(T)$ tel que $0 < |\alpha|_\infty < 1$. Alors $\varpi_u^{(k)}(\alpha)$ est transcendant sur $\mathbb{F}_q(T)$ s'il existe un entier $c \geqslant 1$ tel que $u(n) \neq 0$ pour un nombre infini d'entiers n vérifiant $q^c \nmid n$.*

1. Introduction

Motivated and inspired by Wade's method (see [11,12], and [13]) and also by some considerations on Diophantine approximation in positive characteristic, B. de Mathan proved in [5] a very powerful transcendence criterion, which has many interesting applications in the study of transcendence of formal power series arising from the Carlitz module. It was then generalized independently by L. Denis [6] and by Y. Hellegouarch [7] along two quite different lines. Inspired by the result of B. de Mathan and based on some considerations for automatic sequences, we gave in [15] (see also [14]) a series of transcendence criteria, and obtained, in particular, a new proof of the fact that the sequence of partial quotients of the formal power series of Baum–Sweet is not automatic, a result proved originally by M. Mkaouar in [9] by a quite different method.

Puzzled by the quite evident differences between his criterion and that of Y. Hellegouarch, L. Denis pointed out in [6] that it would be interesting to make a further comparison between these two criteria, neither of which seems to contain the results of the other. In response to this suggestion, but also inspired by all the results mentioned above, we shall present in this work an elementary but rather general criterion in Diophantine approximation, which consists of one real condition, nevertheless covers not only all the criteria cited above, but also many other ones such that the criterion of L.I. Wade [11] and that of S.M. Spencer, Jr. [10], proved originally by Wade's method. As applications, we shall study the transcendence of two families of functions at nonzero algebraic arguments.

Now we fix some basic definitions and notations needed for our later study.

Let K be a fixed field of characteristic $p \geqslant 2$ and $q = p^w$ with $w \geqslant 1$ an integer. We denote by $K[T]$ the integral domain of polynomials in the indeterminate T with coefficients in K. Its field of fractions is noted $K(T)$.

For all $P, Q \in K[T]$ with $Q \neq 0$, we define

$$|P/Q|_\infty := q^{\deg(P)-\deg(Q)}.$$

Finally we denote by $K((T^{-1}))$ the topological completion of $K(T)$ relative to $|\cdot|_\infty$, and by $\mathbf{C}_\infty$ the topological completion of a fixed algebraic closure of $K((T^{-1}))$. It is well known that $\mathbf{C}_\infty$ is topologically complete and algebraically closed.

2. Main result

With the above notations, we have the following criterion:

Theorem 1. *Let $\alpha \in K((T^{-1}))$ be such that there exist $(P_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ two sequences of nonzero elements in $K[T]$ satisfying the following conditions*:

1) *There exists a sequence $(\delta_n)_{n\geqslant 0}$ of positive real numbers such that*

$$|\alpha - P_n/Q_n|_\infty \leqslant \delta_n$$

for all integers $n \geqslant 0$;

2) *For all $(r+1)$-tuples $(A_0, \ldots, A_r)$ of elements in $K[T]$ with $A_r \neq 0$, there exists an infinite set S of integers $n \geqslant r$ such that*

$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{\theta_j(n)}}{Q_{\theta_j(n)}}\right)^{q^j}$$

is different from zero, and for all integers j $(0 \leqslant j \leqslant r)$,

$$\limsup_{\substack{n\in S\\ n\to+\infty}} \frac{|\beta_n|}{\delta_{\theta_j(n)}^{q^j}} = +\infty,$$

where each $\theta_j = (\theta_j(n))_{n\geqslant 0}$ $(0 \leqslant j \leqslant r)$ is a sequence of positive integers increasing to $+\infty$.

Then α is transcendental over $K(T)$.

The first condition is indeed a definition of δ_n. To cover the criterion of Y. Hellegouarch [7], one need only imitate him to introduce a bounded sequence of positive integers $(s_n)_{n\geqslant 0}$ and replace q^j by $q^{s_n+\cdots+s_{n-j+1}}$ in Theorem 1. The proof is essentially the same, so we shall not discuss such a more general, and thus more complicated version here.

3. Proof of Theorem 1

By contradiction, suppose that α were algebraic and of degree $r \geqslant 1$ over $K(T)$. Then the $r+1$ elements $1, \ldots, \alpha^{q^r}$ are linearly dependent over $K(T)$, and thus we can find $A_0, \ldots, A_r$ in $K[T]$, not all zeros, such that

$$\beta := \sum_{j=0}^{r} A_j \alpha^{q^j} = 0.$$

Therefore for all $n \geqslant r$ $(n \in S)$, we have

$$|\beta-\beta_n| \leqslant \max_{0\leqslant j\leqslant r}|A_j|\left|\alpha-\frac{P_{\theta_j(n)}}{Q_{\theta_j(n)}}\right|^{q^j} \leqslant \max_{0\leqslant j\leqslant r}|A_j|\delta_{\theta_j(n)}^{q^j}$$
$$\leqslant \max_{0\leqslant j\leqslant r}|A_j|\cdot \max_{0\leqslant j\leqslant r}\delta_{\theta_j(n)}^{q^j}.$$

But $\beta = 0$, consequently we obtain

$$\limsup_{\substack{n\in S\\ n\to+\infty}} \frac{|\beta_n|}{\max_{0\leqslant j\leqslant r}\delta_{\theta_j(n)}^{q^{\sigma_j}}} \leqslant \max_{0\leqslant j\leqslant r}|A_j|,$$

which contradicts the second condition of Theorem 1. So α is transcendental.

4. Some applications

We fix in this section $K=\mathbb{F}_q$ and $k\geqslant 2$ an integer. Inspired by their classical real counterparts, L.I. Wade introduced in [13] the following two formal power series (with $G\in\mathbb{F}_q[T_1,\ldots,T_\varkappa]$ and $\deg(G)\geqslant 1$)

$$\sum_{n=0}^{+\infty}\frac{1}{G^{k^n}} \quad\text{and}\quad \sum_{n=0}^{+\infty}\frac{1}{G^{n^k}},$$

and showed that the first one is algebraic over $\mathbb{F}_q(T_1,\ldots,T_\varkappa)$ if and only if k is a power of p, and that the second one is always transcendental. His result was then generalized in different directions via different methods by S.M. Spencer, Jr. [10], J.-P. Allouche [3], and V. Laohakosol et al. [8]. Motivated by the above cited work of L.I. Wade, we shall define below two families of functions which take the above two formal power series as special values, and then study their nature at nonzero algebraic arguments. For simplicity, here we shall only concentrate our attention on the case of one indeterminate T, although our results also hold for the general case. Indeed, one need only use an argument developed by J.-P. Allouche in [3] (see also [2]) to pass from the one variable case to the multi-variables case.

Let $u=(u(n))_{n\geqslant 0}$ be a sequence with values in $\mathbb{F}_q$. Define

$$\kappa_u^{(k)} := \sum_{n=0}^{+\infty}u(n)t^{k^n}, \quad\text{and}\quad \varpi_u^{(k)} := \sum_{n=0}^{+\infty}u(n)t^{n^k}.$$

One can note that if the sequence u is not ultimately zero, then the convergence radii of $\kappa_u^{(k)}$ and $\varpi_u^{(k)}$ are both equal to 1.

As applications of Theorem 1, we shall show the following two results:

Theorem 2. *Let $u=(u(n))_{n\geqslant 0}$ be a sequence in $\mathbb{F}_q$ which is not ultimately zero. Let $\alpha\in\mathbf{C}_\infty$ be algebraic over $\mathbb{F}_q(T)$ satisfying $0<|\alpha|_\infty<1$. Then $\kappa_u^{(k)}(\alpha)$ is algebraic over $\mathbb{F}_q(T)$ if and only if k is a power of p and u is ultimately periodic.*

Theorem 3. *Let $u=(u(n))_{n\geqslant 0}$ be a sequence in $\mathbb{F}_q$ which is not ultimately zero. Let $\alpha\in\mathbf{C}_\infty$ be algebraic over $\mathbb{F}_q(T)$ such that $0<|\alpha|_\infty<1$. Then $\varpi_u^{(k)}(\alpha)$ is transcendental over $\mathbb{F}_q(T)$ if there exists an integer $c\geqslant 1$ such that $u(n)\neq 0$ holds for an infinite number of integers n satisfying $q^c\nmid n$.*

Theorem 3 may fail to hold if u does not satisfy the imposed condition which is surely unnecessary. For example, if $u(n)=0$ for all integers $n\geqslant 0$ which are not powers of p, then we fall into Theorem 2, by which we know that $\varpi_u^{(k)}(\alpha)$ is algebraic over $\mathbb{F}_q(T)$ if and only if the sequence $(u(p^j))_{j\geqslant 0}$ is ultimately periodic.

5. Proofs of Theorem 2 and Theorem 3 (sketches)

First of all, we note that α is transcendental over $\mathbb{F}_q$. Indeed if α were algebraic over $\mathbb{F}_q$, then $\alpha\in\mathbb{F}_{q^\ell}$ for some integer $\ell\geqslant 1$, and thus $|\alpha|_\infty=1$. As a result, we obtain that T is algebraic over $\mathbb{F}_q(\alpha)$ for α, T are algebraically dependent over $\mathbb{F}_q$, while they are both transcendental over $\mathbb{F}_q$.

Second, let $v = (v(n))_{n\geqslant 0}$ be a sequence not ultimately zero in $\mathbb{F}_q$, and set

$$f = \sum_{n=0}^{+\infty} v(n)t^n.$$

Now that α and T are algebraically dependent, the transcendence over $\mathbb{F}_q(\alpha)$ and that over $\mathbb{F}_q(T)$ are equivalent, for they are both equivalent to the transcendence over $\mathbb{F}_q(\alpha, T)$ (consider for example the transcendence degrees of the corresponding extensions). Note also that $f(\alpha)$ is algebraic over $\mathbb{F}_q(\alpha)$ if and only if $f(T^{-1})$ is algebraic over $\mathbb{F}_q(T)$, for α is transcendental over $\mathbb{F}_q$. Consequently $f(\alpha)$ is algebraic over $\mathbb{F}_q(T)$ if and only if $f(T^{-1})$ is. Indeed, if $f(\alpha)$ is algebraic over $\mathbb{F}_q(T)$, then it is also algebraic over $\mathbb{F}_q(\alpha)$, and thus $f(T^{-1})$ is algebraic over $\mathbb{F}_q(T)$. Conversely, if $f(T^{-1})$ is algebraic over $\mathbb{F}_q(T)$, then $f(\alpha)$ is algebraic over $\mathbb{F}_q(\alpha)$, so over $\mathbb{F}_q(T)$. Therefore, to prove our theorems, we need only treat the case $\alpha = T^{-1}$.

If k is a power of p, then we drop on a problem already solved by C. Cadic in [4] (see also [16]) via automata theory. Of course, this result can also be treated rather easily and directly by definition.

Now we show by Theorem 1 that if k is not a power of p, then $\kappa_u^{(k)}(T^{-1})$ is transcendental over $\mathbb{F}_q(T)$. Let $r \geqslant 1$ be an integer, and let $A_0, \dots, A_r$ be elements in $\mathbb{F}_q[T]$ with $A_r \neq 0$. For each integer j $(0 \leqslant j \leqslant r)$, define the integer ρ_j by

$$\frac{q^{r-j}}{k} < k^{\rho_j} \leqslant q^{r-j}, \tag{1}$$

and set $\theta_j(n) = n + \rho_j$, for all integers $n \geqslant 0$. Note here that $\rho_r = 0$, and that the equality in (1) cannot hold for $j < r$ since k is not a power of p.

Let $n \geqslant 0$ be an integer. Define

$$P_n := \sum_{i=0}^{n} u(i)T^{k^n - k^i}, \quad Q_n := T^{k^n}, \quad \text{and} \quad \delta_n := q^{-k^{n+1}},$$

$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{\theta_j(n)}}{Q_{\theta_j(n)}}\right)^{q^j}, \quad \text{and} \quad C_n := T^{k^n q^r}\beta_n.$$

Then $|\kappa_u^{(k)}(T^{-1}) - P_n/Q_n|_\infty \leqslant \delta_n$, $C_n \in \mathbb{F}_q[T]$, and $C_n \equiv A_r u(n) \pmod{T^{\mu_n}}$ with

$$\mu_n := \min_{0\leqslant j<r} \left(k^n q^r - k^{n+\rho_j} q^j, k^n q^r - k^{n-1} q^r\right).$$

Let S be the set of all integers n such that $u(n) \neq 0$ and $\mu_n > \deg(A_r)$. Then S is infinite, and for all $n \in S$, we have $C_n \neq 0$, and thus

$$|\beta_n|_\infty = \frac{|C_n|_\infty}{|T^{k^n q^r}|_\infty} \geqslant \frac{1}{|T^{k^n q^r}|_\infty} = q^{-k^n q^r},$$

which implies that $\kappa_u^{(k)}(T^{-1})$ is transcendental by Theorem 1.

For the proof of Theorem 3, we apply Theorem 1 with q^{kc} in the place of q.

Let $r \geqslant 1$ be an integer, and let $A_0, \dots, A_r$ be elements in $\mathbb{F}_q[T]$ with $A_r \neq 0$. By using the Cartier operators for $\mathbb{F}_q((T^{-1}))$, we can, just as in [1], suppose without loss of generality $A_0 \neq 0$. For all integers $n \geqslant 0$, define

$$P_n := \sum_{i=0}^{n} u(i)T^{n^k - i^k}, \quad Q_n := T^{n^k}, \quad \text{and} \quad \delta_n := q^{-(n+1)^k},$$

$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{\theta_j(n)}}{Q_{\theta_j(n)}}\right)^{q^{jkc}}, \quad \text{and} \quad C_n := T^{n^k}\beta_n,$$

where $\theta_j(n) := \lfloor n/q^{jc} \rfloor$ is the greatest integer $\leqslant n/q^{jc}$. Then $C_n \in \mathbb{F}_q[T]$, and

$$\left|\varpi_u^{(k)}\left(T^{-1}\right) - P_n/Q_n\right|_\infty \leqslant \delta_n.$$

Let S be the set of all $n \geqslant 0$ such that $u(n) \neq 0$ and $q^c \nmid n$. By hypothesis S is infinite. Set $\eta_j(n) := n - q^{jc}\theta_j(n)$, for all $j, n \geqslant 0$ $(0 \leqslant j \leqslant r)$. Then $\eta_0(n) = 0$, and for all $n \in S$, we have $1 \leqslant \eta_j(n) < q^{jc}$ $(1 \leqslant j \leqslant r)$.

Set

$$\mu_n := \min_{1 \leqslant j \leqslant r} \left(n^k - \left(\theta_j(n)\right)^k q^{jkc}, n^k - (n-1)^k\right).$$

Then $\mu_n \to +\infty$ with $n \in S$, and for all $n \in S$ such that $\mu_n > \deg(A_0)$, we have

$$C_n \equiv A_0 u(n) \left(\operatorname{mod} T^{\mu_n}\right).$$

Consequently $C_n \neq 0$, and thus

$$|\beta_n|_\infty = \frac{|C_n|_\infty}{|T^{n^k}|_\infty} \geqslant \frac{1}{|T^{n^k}|_\infty} = q^{-n^k},$$

which yields the desired result with the help of Theorem 1.

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