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Functional Analysis

Weighted pseudo almost periodic functions and applications

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Abstract

This Note introduces a new class of functions called weighted pseudo almost periodic functions, which generalize in a natural fashion the classical pseudo almost periodic functions due to C. Zhang. Properties of those weighted pseudo almost periodic functions are discussed including a composition result of weighted pseudo almost periodic functions, which plays a crucial role for the solvability of some weighted pseudo almost periodic semilinear differential and partial differential equations. ***To cite this article: T. Diagana, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Les fonctions pseudo presque périodiques avec poids et applications. Dans cette Note on introduit une nouvelle classe de fonctions que nous appelerons fonctions pseudo presque périodiques avec poids, lesquelles généralisent de façon naturelle les fonctions pseudo presque périodiques que C. Zhang avait introduit dans la littérature il y a une dizaine d'années. Les propriétés de telles fonctions sont étudiées, en particulier, un résultat sur la composition de fonctions pseudo presque périodiques avec poids est obtenu, lequel sert de support lorsqu'on étudie les solutions pseudo presque périodiques avec poids de certaines équations différentielles ou équations aux dérivées partielles sémi-linéaires. ***Pour citer cet article : T. Diagana, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Dans cette Note nous étendons la classe des fonctions pseudo presque périodiques introduites par Zhang [5–7]. Pour cela, l'idée principale consiste a mettre un poids $\rho : \mathbb{R} \mapsto (0, \infty)$ (une fonction localement intégrable sur tout $\mathbb{R}$) sur la composante ergodique apparaissant dans la définition de Zhang (voir (2)) et ainsi obtenir l'espace a poids, $PAP_0(\mathbb{X}, \rho)$ (voir (1)). De cette façon, une fonction pseudo presque périodique de poids ρ, f, apparaitra comme étant une perturbation d'une fonction presque périodique au sens de H. Bohr par une fonction de $PAP_0(\mathbb{X}, \rho)$, c'est à dire, $f = g + \phi$ où $g \in AP(\mathbb{X})$ et $\phi \in PAP_0(\mathbb{X}, \rho)$ (voir Definition 2.3). Entre autres, il est montré que si deux poids ρ_1 et ρ_2 sont équivalents (voir Definition 3.1), alors $PAP_0(\mathbb{X}, \rho_1) = PAP_0(\mathbb{X}, \rho_2)$. En particulier, si le poids ρ est borné et si $\inf_{x \in \mathbb{R}} \rho(x) > 0$, alors $PAP_0(\mathbb{X}) = PAP_0(\mathbb{X}, \rho)$. Aussi, un résultat sur la composition de fonctions pseudo presque

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périodiques avec poids est obtenu, lequel sert de support lorsqu'on étudie les solutions pseudo presque périodiques avec poids de certaines équations différentielles ou équations aux dérivées partielles sémi-linéaires (voir Theorem 4.1 pour le cas d'une équation différentielle sémi-linéaire).

1. Introduction

The concept of *pseudo* almost periodicity, which is the central question in this Note was introduced in the literature in the early nineties by Zhang [5–7], as a natural generalization of the classical *almost periodicity* in the sense of H. Bohr. One should also point out that the notion of pseudo almost periodicity implements another existing generalization of the H. Bohr's almost periodicity, the so-called *asymptotically* almost periodicity due to Fréchet.

In this Note we introduce a new class of functions called *weighted* pseudo almost periodic functions, which generalize in a natural manner the Zhang's pseudo almost periodic functions. To do so, our main idea consists of enlarging the so-called *ergodic* component, utilized in the Zhang's definition of pseudo almost periodicity [2–5], with the help of a *weighted measure* $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$, where $\rho : \mathbb{R} \mapsto (0, \infty)$ is a *locally integrable* function over $\mathbb{R}$. Note that ρ given above is commonly called *weight*.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be Banach spaces and let $(BC(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty)$ denote the Banach space of all bounded continuous functions. Throughout the rest of this paper, $\mathbb{U}$ stands for the collection of functions (weights) $\rho : \mathbb{R} \mapsto (0, \infty)$, which are locally integrable over $\mathbb{R}$ with $\rho > 0$ (a.e.). Let $\mathbb{U}_{\mathbb{B}}$ denote the set of all $\rho \in \mathbb{U}$ such that ρ is bounded with $\inf_{x\in\mathbb{R}} \rho(x) > 0$.

From now on, if $\rho \in \mathbb{U}$ and for $T > 0$, we then set: $\mu(T, \rho) := \int_{-T}^{T} \rho(x)\,\mathrm{d}x$.

Definition 2.1. A strongly continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $T_0(\varepsilon) > 0$ such that every interval of length $T_0(\varepsilon)$ contains a number τ with the property that $\|f(t+\tau) - f(t)\| < \varepsilon$ for each $t \in \mathbb{R}$. The collection of those functions is denoted by $AP(\mathbb{X})$.

Definition 2.2. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $\phi \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $T_0(\varepsilon)$ such that every interval of length $T_0(\varepsilon)$ contains a number τ with the property that $\|F(t+\tau, \phi) - F(t, \phi)\| < \varepsilon$ for each $t \in \mathbb{R}$, $\phi \in K$. The collection of those functions is denoted by $AP(\mathbb{Y}, \mathbb{X})$.

Let $\rho \in \mathbb{U}$. Define

$$PAP_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) \colon \lim_{T\to\infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \|f(\sigma)\| \rho(\sigma)\,\mathrm{d}\sigma = 0 \right\}. \tag{1}$$

Clearly, when the weight $\rho(x) = 1$ for each $x \in \mathbb{R}$, one retrieves the so-called ergodic space of Zhang, that is, $PAP_0(\mathbb{X})$, defined by

$$PAP_0(\mathbb{X}) = \left\{ f \in BC(\mathbb{R}, \mathbb{X}) \colon \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(\sigma)\|\,\mathrm{d}\sigma = 0 \right\}. \tag{2}$$

In the same way, we define $PAP_0(\mathbb{Y}, \mathbb{X}, \rho)$ as the collection of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that $F(\cdot, \phi)$ is bounded for each $\phi \in \mathbb{Y}$ and

$$\lim_{T\to\infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \|F(s, \phi)\| \rho(s)\,\mathrm{d}s = 0, \quad \text{uniformly in } \phi \in \mathbb{Y}.$$

We are now ready to define weighted pseudo almost periodic functions.

Definition 2.3. A function $f \in \mathrm{BC}(\mathbb{R}, \mathbb{X})$ is called weighted pseudo almost periodic (or ρ-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$. The collection of such functions will be denoted by $PAP(\mathbb{X}, \rho)$.

Remark 1. The decomposition of a ρ-pseudo almost periodic function $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$, is unique. This is mainly based upon the fact that $g(\mathbb{R}) \subset \overline{f(\mathbb{R})}$. Hence, $PAP(\mathbb{X}, \rho) = AP(\mathbb{X}) \oplus PAP_0(\mathbb{X}, \rho)$.

Definition 2.4. A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called weighted pseudo almost periodic (or ρ-pseudo almost periodic) in $t \in \mathbb{R}$ uniformly in $\phi \in \mathbb{Y}$ if it can be expressed as $F = G + \Phi$, where $G \in AP(\mathbb{Y}, \mathbb{X})$ and $\phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \rho)$. The collection of such functions will be denoted by $PAP(Y, \mathbb{X}, \rho)$.

3. Properties of weighted pseudo almost periodic functions

Definition 3.1. Let $\rho_1, \rho_2 \in \mathbb{U}$. One says that ρ_1 is equivalent to ρ_2 or $\rho_1 \sim \rho_2$ whenever $(\rho_1/\rho_2) \in \mathbb{U}_{\mathbb{B}}$.

Theorem 3.2. *Let $\rho_1, \rho_2 \in \mathbb{U}$. If $\rho_1 \sim \rho_2$, then $PAP(\mathbb{X}, \rho_1) = PAP(\mathbb{X}, \rho_2)$.*

An immediate consequence of Theorem 3.2 is the next corollary, which enables us to connect the Zhang's space $PAP(\mathbb{X}) = AP(\mathbb{X}) \oplus PAP_0(\mathbb{X})$ with a weighted pseudo almost periodic class $PAP(\mathbb{X}, \rho)$.

Corollary 3.3. *If $\rho \in \mathbb{U}_{\mathbb{B}}$, then $PAP(\mathbb{X}, \rho) = PAP(\mathbb{X})$.*

Theorem 3.4. *Let $f \in PAP(\mathbb{Y}, \mathbb{X}, \rho)$ satisfying the Lipschitz condition*

$$\left\| f(t,u) - f(t,v) \right\| \leqslant K \|x - y\|_{\mathbb{Y}} \quad \text{for all } u, v \in \mathbb{Y},\ t \in \mathbb{R}. \tag{3}$$

If $h \in PAP(\mathbb{Y}, \rho)$, then $f(\cdot, h(\cdot)) \in PAP(\mathbb{X}, \rho)$.

Theorem 3.4 generalize known composition results of pseudo almost periodic functions, in particular those given in both [1] and [6].

Corollary 3.5. *Let $\rho_1, \rho_2 \in \mathbb{U}$. Suppose that $\rho_1 \sim \rho_2$, $f \in PAP(\mathbb{Y}, \mathbb{X}, \rho_1)$, and f satisfies* (3). *If $h \in PAP(\mathbb{Y}, \rho_2)$, then $f(\cdot, h(\cdot)) \in PAP(\mathbb{X}, \rho_1 + \rho_2)$.*

4. Application

Let $\rho \in \mathbb{U}$. This section is devoted to the existence of a ρ-pseudo almost periodic solution to the abstract differential equation

$$u'(t) = Au(t) + f\big(t, u(t)\big), \quad t \in \mathbb{R}, \tag{4}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup on a Banach space $\mathbb{X}$ and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is a ρ-pseudo almost periodic function in the first variable uniformly in the second variable.

To deal with the existence and uniqueness of ρ-pseudo almost periodic solutions to (4), the following assumptions will be made:

(H.1) The function $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$, $(t, u) \mapsto f(t, u)$ is ρ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$, and f satisfies (3).

(H.2) The operator A is the infinitesimal generator of an exponentially stable c_0-semigroup; thus there exist constants $M > 0$ and $\omega > 0$ such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for each $t \geqslant 0$.

(H.3) Let $\omega > 0$ be the constant appearing in (H.2). Setting

$$P(\omega) := \sup_{T>0} \left\{ \int_{-T}^{T} \mathrm{e}^{-\omega(T+t)} \rho(t)\, \mathrm{d}t \right\}, \quad \text{we suppose that } P(\omega) < \infty.$$

Theorem 4.1. *Suppose that assumptions* (H.1)–(H.3) *above hold and that* $\lim_{T\to\infty}\mu(T,\rho)=\infty$. *Then* (4) *has a unique* ρ*-pseudo almost periodic solution whenever* $K<\frac{\omega}{M}$.

Remark 2. Note that in the particular case when $\rho\in\mathbb{U}_{\mathbb{B}}$, that is, $PAP(\mathbb{X},\rho)=PAP(\mathbb{X})$ by Corollary 3.3, we retrieve the 'non-weighted' situation, since assumption (H.3) is always achieved in that event. This means that weighted pseudo almost periodic functions are good generalizations of the Zhang's pseudo almost periodic functions.

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