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C. R. Acad. Sci. Paris, Ser. I 343 (2006) 747–749

COMPTES RENDUS MATHEMATIQUE

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Analytic Geometry

On the Brieskorn (a, b)-module of an isolated hypersurface singularity

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Received 30 August 2006; accepted 26 September 2006

Available online 20 November 2006

Presented by Bernard Malgrange

Abstract

For a germ g of holomorphic function with an isolated singularity at the origin of $\mathbb{C}^n$ we show that there is a pole for the meromorphic continuation of the distribution $\frac{1}{\Gamma(\lambda)}.|g|^{2\lambda}\bar{g}^{-n}$ at $\lambda = -n - \alpha$ where α is the smallest root in its class modulo $\mathbb{Z}$ of the reduced Bernstein–Sato polynomial of g. This rather 'unexpected' result is a consequence of the fact that the self-duality of the Brieskorn (a, b)-module E_g associated to g exchanges the biggest simple pole sub-(a, b)-module of E_g with the saturation of E_g by $b^{-1}a$. ***To cite this article: D. Barlet, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

A propos du (a, b)-module de Brieskorn d'une fonction à singularité isolée. Pour un germe g de fonction holomorphe à singularité isolée à l'origine de $\mathbb{C}^n$ nous montrons que le prolongement méromorphe de la distribution $\frac{1}{\Gamma(\lambda)}.|g|^{2\lambda}\bar{g}^{-n}$ admet un pôle en $\lambda = -n - \alpha$ où α est la plus petite racine dans sa classe modulo $\mathbb{Z}$ du polynôme réduit de Bernstein–Sato de g. Ce résultat assez « inattendu » est conséquence du fait que l'auto-dualité du (a, b)-module de Brieskorn E_g associé à g échange le plus grand sous-module à pôle simple de E_g avec le saturé de E_g par $b^{-1}a$. ***Pour citer cet article : D. Barlet, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Let $\tilde{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic function with an isolated singularity. Denote by $g : X \to D$ a Milnor representative of $\tilde{g}$. Let b_g be the reduced[1] Bernstein–Sato polynomial of g. Let α be the biggest root of b_g in its class modulo $\mathbb{Z}$. A classical question is whether for $j \in \mathbb{N}$ big enough, the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|g|^{2\lambda}\bar{g}^{-j}\Box$ has a pole at $\lambda = \alpha$. The following result suggests that, maybe, this question is not the right one:

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[1] If b is the usual Bernstein–Sato polynomial of g, it is defined by the formula $(s+1).b_g(s) = b(s)$. Then b_g is the minimal polynomial of the action of $-b^{-1}a$ acting on $\tilde{E}_g/b.\tilde{E}_g$ where $\tilde{E}_g$ is the saturation of E_g by $b^{-1}a$; see [3].

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.10.001

Theorem 1. *Let α be the* smallest[2] *root of b_g in its class modulo $\mathbb{Z}$, and let d be its multiplicity* (*as a root of b_g*)*. Then the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|g|^{2\lambda}\bar{g}^{-n}$ on X has a pole of order $\geqslant d$ at $-n-\alpha$.*

Remarks.

(1) In general it is not clear that $-n-\alpha$ is a root of b_g. But, of course, the previous theorem implies that there exists at least d roots of b_g (counting multiplicities) equal to $-\alpha$ *modulo* $\mathbb{Z}$ which are bigger than $-n-\alpha$. If $-n-\alpha \in [-1, 0[$ then there is no choice: $-n-\alpha$ is a root of multiplicity $\geqslant d$ of b_g.

(2) This result gives, in term of the Bernstein–Sato polynomial b_g, a precise value where we know that a pole appears in the class $[\beta]$ modulo $\mathbb{Z}$ of a root β of b_g. But the pole which is given is not at the biggest root of b_g in this class but at the biggest root of the polynomial $b_g^*(z) := b_g(-n-z)$ in this class!
A clear reason for that is given in the proof: the dual Bernstein–Sato polynomial b_g^* is the minimal polynomial of $-b^{-1}a$ acting on $F/b.F$ where F is the biggest simple pole sub-(a,b)-module of the Brieskorn (a,b)-module E associated to g. So it lies in the lattice given by *holomorphic* forms.
On the contrary, b_g is the minimal polynomial of $-b^{-1}a$ acting on $\tilde{E}/b\tilde{E}$ where $\tilde{E}$ is the saturation of E by $b^{-1}a$, or, in other words, the minimal simple pole (a,b)-module containing E. So, if E is not a simple pole (a,b)-module, elements in $\tilde{E}$ are not always representable in the holomorphic lattice, and so we may need some power of g as denominators. Also this may introduce integral shifts for the poles.

(3) The case where E is a simple pole (a,b)-module (that is to say when we have $F = E = \tilde{E}$) corresponds to a quasi-homogeneous g, with a suitable choice of coordinates. In this case we have $b_g^* = b_g$, so $-n-\alpha$ is the smallest root of b_g in its class modulo $\mathbb{Z}$. However, of course, this case was already known long ago.

Let me recall some basic facts for the convenience of the reader:

(1) An (a,b)-module E is a free finite type module over the ring $\mathbb{C}[[b]]$ with an $\mathbb{C}$-linear endomorphism $a : E \to E$ such that a is continuous for the b-adic topology and satisfies $ab - ba = b^2$. For instance, for any $\delta \in \mathbb{C}$, endowed with a defined by $a.e = \delta.b.e$ the rank one $\mathbb{C}[[b]]$-module $\mathbb{C}[[b]].e$ is an (a,b)-module. For $\delta = 1$ it is easy to identify E_1 with $\mathbb{C}[[z]]$ where b is the primitive without constant and a is the multiplication by z.

(2) For any germ g of holomorphic function with an isolated singularity at the origin of $\mathbb{C}^n$ the formal completion in g of the "usual" Brieskorn module $\Omega_0^n / dg \wedge d\Omega_0^{n-2}$ (see [8]) of g is an (a,b)-module.[3] It will be denoted E_g (see [3]). This Brieskorn (a,b)-module is regular, that means that its saturation by $b^{-1}a$ is again a finite type $\mathbb{C}[[b]]$-module.

(3) We say that the (a,b)-module E has a simple pole[4] when $a.E \subset b.E$. Of course a simple pole (a,b)-module is regular. Any regular (a,b)-module E contains a biggest sub-(a,b)-module with simple pole (see [6]).

(4) For an (a,b)-module E note $\check{E}$ the (a,b)-module obtained by changing a and b in $-a$ and $-b$. For two (a,b)-modules E, F let $\mathrm{Hom}_{a,b}(E,F)$ be the (a,b)-module obtained as follows (see [4] and compare with the tensor product introduced in [5]): as an $\mathbb{C}[[b]]$-module it is equal to $\mathrm{Hom}_{\mathbb{C}[[b]]}(E,F)$. The endomorphism a is given on it by

$$(a.\varphi)(x) = a_F.\varphi(x) - \varphi(a_E.x) \quad \forall x \in E \ \varphi \in \mathrm{Hom}_{\mathbb{C}[[b]]}(E,F).$$

(5) Another special property of the (a,b)-module E_g is that we have a canonical isomorphism of (a,b)-modules[5] (see [7])

$$\check{E}_g \simeq \mathrm{Hom}_{a,b}(E_g, E_n)$$

where $E_n = E_\delta$ for $\delta = n$ (see example in (1) above).

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[2] Recall that we are dealing with negative numbers.
[3] With $a := \times g$ and $b := dg \wedge d^{-1}$.
[4] This corresponds to the usual notion of simple pole for a singular point of an ordinary holomorphic differential equation.
[5] Corresponding to the microlocal duality.

Proposition 2. *Assume that E is a regular (a,b)-module such that for some $\delta \in \mathbb{C}$ we have an isomorphism of (a,b)-module $\check{E} \simeq \mathrm{Hom}_{a,b}(E_g, E_\delta)$. Let F the biggest simple pole sub-(a,b)-module of E and $\tilde{E}$ the saturation of E by $b^{-1}a$. Then the are (a,b)-modules isomorphisms*

$$\check{\tilde{E}} \to \mathrm{Hom}_{a,b}(F, E_\delta) \quad \text{and} \quad \check{F} \to \mathrm{Hom}_{a,b}(\tilde{E}, E_\delta).$$

Definition 3. Let E be a regular (a,b)-module, and let F be its biggest simple pole sub-(a,b)-module. We shall call *dual Bernstein–Sato polynomial* of E, denoted by b_E^*, the minimal polynomial of the action of $-b^{-1}a$ on the finite dimensional $\mathbb{C}$-vector space $F/b.F$.

Remark. Let δ be a given complex number, and assume that the (a,b)-module E is equipped with an (a,b)-linear isomorphism $\kappa : \check{E} \to \mathrm{Hom}_{a,b}(E, E_\delta)$. Then we have $b_E^*(z) = (-1)^r.b_E(-\delta - z)$ where $r := \deg(b_E)$, since $b^{-1}a$ acts on the same way on E and $\check{E}$. So, for the Brieskorn (a,b)-module of a germ of an holomorphic function g with an isolated singularity at the origin of $\mathbb{C}^n$ the dual Bernstein polynomial of E_g is the dual Bernstein–Sato polynomial of g as defined before (up to a sign). Using Malgrange's positivity theorem (see [10]) it is easy to show that the roots of b_g^* are strictly negative. This gives, using [9], the fact that the roots of b_g are contained in $]-n, 0[$.

Sketch of proof of the theorem. The only new point for this proof, compared to [1] and [2], is the following:

In a simple pole (a,b)-module F, if a spectral value β of multiplicity d for the action of $b^{-1}.a$ on F/bF, is minimal in its class modulo $\mathbb{Z}$, there exists elements $e_1, \ldots, e_d$ in F, giving a Jordan block of size d for $b^{-1}a$ acting on F/bF, and such that they satisfy in F the relations $a.e_j = \beta.b.e_j + b.e_{j-1}$, $\forall j \in [1,d]$ with the convention $e_0 = 0$ (see [3]).

This enable us, using the standard technics of [1], to build up $(n-1)$-holomorphic forms $\omega_1, \ldots, \omega_d$ in a neighbourghood of the origin in $\mathbb{C}^n$, such that $d\omega_j = \beta.\frac{dg}{g} \wedge \omega_j + \frac{dg}{g} \wedge \omega_{j-1}$, $\forall j \in [1,d]$ (with the convention $\omega_0 = 0$), which induce a Jordan block of size d in $H^{n-1}(F, \mathbb{C})$ where F is the Milnor fiber of g, for the eigenvalue $\exp(-2i\pi.\beta)$ of the monodromy.

So we avoid in this way the integral shift coming from the use of a lattice which may be not contained in the one given by holomorphic forms and we can realize the pole of our statement for $\lambda = -\beta$, using the same strategy than in [1] for eigenvalues $\neq 1$ and [2] for the eigenvalue 1. $\square$

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