



Differential Geometry/Calculus of Variations

Positive scalar curvature in $\dim \geq 8$

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Abstract

We announce a first series of new results and techniques extending the scope of applications of minimal hypersurfaces in scalar curvature geometry. For instance, the restriction to dimensions ≤ 7 which arises from subtle analytic problems in higher dimensions is entirely removed. *To cite this article: J. Lohkamp, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Courbure scalaire positive en dimension ≥ 8 . Nous annonçons une suite des résultats et techniques nouveaux qui permet d'étendre les domaines d'application des hypersurfaces minimaux en géométrie de courbure scalaire. Par exemple, la restriction aux dimensions ≤ 7 qui émerge d'un problème analytique subtil en dimensions plus grandes est éliminée complètement. *Pour citer cet article : J. Lohkamp, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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1. Introduction and statement of results

Scalar curvature is presently studied most successfully in the case where the underlying manifold is *Spin*: surgery techniques and the index theorem provide a rather direct and interactive link to topology (cf. [2,3]).

However getting a more geometric insight, and handling the general case including *Non-Spin* manifolds, requires another approach: in the late 1970s minimal hypersurfaces within the manifold under consideration (cf. [9,10]) turned out to be a good candidate: they have the remarkable property to gather efficiently positive scalar curvature from the ambient manifold while the dimension declines. This can be iterated until one reaches a lower dimensional geometry/topology that is well understood (for instance that of surfaces); then one can analyze the induced metric to derive information for the original scalar curvature geometry.

Being based on geometric measure (and regularity) theory the minimal hypersurface approach was soon bound to run into trouble. Most seriously the appearance of still hardly understood singularities of area minimizing hypersurfaces in dimensions ≥ 8 made the study of scalar curvature in higher dimensions basically impracticable.

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In this set of papers our main focus will be on the development of some natural techniques that allow us to bypass this problem – by some kind of coarse regularization results – without losing the information encoded in the singular hypersurfaces.

Conceptually, all the new tools are designed to assemble data like *minimal hypersurface + functions solving certain elliptic equations + further minimal submanifolds* to canonical objects which obey *compactness theorems* arising from combinations of geometric and elliptic compactness results. Then one can consider extremal objects which are usually area minimizing cones and reduce the problems by induction.

Our first theorem (derived in such a way) says: after deleting a carefully chosen neighborhood of the singular set within the area minimizer and doubling the resulting manifold one can show that this is a sufficiently good substitute for the original hypersurface.

In order to state it precisely we consider a closed n -dimensional Riemannian manifold (M^n, g) , $n \geq 8$, with *positive* scalar curvature ($scal > 0$), a given homology class $\alpha \in H_{n-1}(M, \mathbb{Z})$.

Classical geometric measure theory provides us with an area minimizing hypersurface H^{n-1} in (M^n, g) representing α which in general (and of course we assume this is the case) contains a compact singular set Σ^{n-8} of Hausdorff-dimension $\leq n - 8$, $\emptyset \neq \Sigma^{n-8} \subset H^{n-1}$. Then we have cf. [1,4]:

Theorem 1. *For any $\varepsilon > 0$ there is a neighborhood $V_\varepsilon \subset \varepsilon$ -neighborhood of Σ^{n-8} in H^{n-1} such that the smooth doubling $H^{n-1} \setminus V_\varepsilon \cup_\sim H^{n-1} \setminus V_\varepsilon$, where \sim means gluing along ∂V_ε , admits a smooth metric g_ε with $scal(g_\varepsilon) > 0$.*

For the proof the codimension of Σ^{n-8} actually used is not 7 but only > 2 : the geometric outcome could therefore be understood as a generalization of the $scal > 0$ -preserving $codim \geq 3$ -surgeries in [2] and [11] but not along a smooth submanifold within a manifold with $scal > 0$ but along Σ^{n-8} in a space whose first eigenvalue for (a scaling invariant refinement of) the conformal Laplacian is positive.

We also set up some more analytic regularization technique which allows us in several situations to assume that the singular minimal hypersurface is actually regular. Specifically we introduce *parametric minimal hypersurfaces with obstacles* as a construction tool in scalar curvature geometry. Formally, take two (for now) smooth compact and cobordant but not necessarily connected submanifolds M_1^m, M_2^m and the cobordism W^{m+1} equipped with some Riemannian metric.

Definition. An area minimizing current \mathcal{T} in W^{m+1} homologous to M_1^m (and thus to M_2^m) is called an area minimizer with obstacles M_1^m and M_2^m .

The context is as follows: start with a free minimizer or a minimizer with already some obstacles. Then the obstacles can be enhanced such that the new minimizer with obstacle is quite regular (cf. [5]):

Theorem 2. *Let V^{n-1} be the unique area minimizer with obstacles and within some homology class of some compact orientable (M^n, g_M) .*

Assume that all obstacles are in contact on one-side of V^{n-1} , then we can place additional obstacles on the same side and find a new C^1 -smooth area minimizer \mathcal{V}^{n-1} arbitrarily near to V^{n-1} in Hausdorff-topology within the same homology class with both classes of objects as obstacles.

After some additional smoothing we get a *smooth* hypersurface (with almost minimal volume) with *positive mean curvature* arbitrarily close to the original singular minimizer. Theorem 2 (which also enters in the proof of Theorem 1) enhances the versatility of the approach considerably since it allows us to combine minimal hypersurfaces techniques with local deformations, surgeries etc.

The techniques developed in the proofs can probably be modified and/or extended to handle the question whether a singular minimal hypersurface could be perturbed into a smooth one at least in the so-called strictly stable case (although this does not mean that the answer should be affirmative). However, this also implies that such a (speculative) regularization result would not provide a more direct approach.

The main application of Theorem 1 (partially in junction with Theorem 2) we want to state in this announcement is the extension of the obstruction theory for $scal > 0$ on large manifolds to a broader class of spaces and to *arbitrary dimensions*:

In particular the *non-existence* of $scal > 0$ -metrics on enlargeable manifolds (e.g. $T^n \# N^n$) [6] or the incompatibility of largeness in a metric sense with positive scalar curvature and more generally lower scalar curvature bounds which can be studied using e.g. the brane action which serves as a substitute for the area functional.

For non-compact manifolds one gets results for sufficiently tame ends, i.e. product like or asymptotically flat resp. hyperbolic ends. In particular, we can use this result (and some new deformation tools) to derive short geometric proofs of the general positive mass conjectures in every dimension [7] and [8] which extend to more advanced versions (e.g. with certain non-asymptotically flat complete ends) and to higher dimensional Penrose inequalities.

2. Sketch of some arguments

We mainly comment on Theorem 1: the argument can be split into some main steps although these are interlaced since we use induction arguments in many places which presume the corresponding *global* results in lower dimensions.

First one checks that from the area minimizing property

$$\lambda_0 := \inf_{f \neq 0, \text{ smooth, supp } f \subset H \setminus \Sigma} \frac{\int_{H \setminus \Sigma} |\nabla f|^2 + \frac{n-2}{4(n-1)} scal_H f^2}{\int_{H \setminus \Sigma} |A|^2 \cdot f^2} > 1/4$$

and we can find a smooth function positive (although possibly not-integrable) function u_0 on $H \setminus \Sigma$ with

$$-\Delta u_0 + \frac{n-2}{4(n-1)} scal_H u_0 = \lambda_0 \cdot |A|^2 \cdot u_0.$$

There is an essential difference to the classical case: the appearance of $|A|^2$ on the right hand side will make parts of the problem scaling invariant. Different from the case of a closed manifold there will usually be many solutions for this problem and they will not depend smoothly on varying data. However in our case we choose particular solutions which are in a sense *minimal*. This is a property inherited by the tangent cones and due to that scaling invariance we find the same type of equation and solutions on the cones. The point is that these solutions have sort of an invariance along the cone direction and thus they can be understood by induction. This analytic scheme is built up in [4].

The conformally deformed metric $u_0^{4/n-2} \cdot g_H$ (g_H is the metric induced on H from (M^n, g)) has $scal > 0$ (actually $scal \geq 0$ but one can deform it to $scal > 0$) as in the classical case:

$$-4(n-1)/(n-2) \cdot \Delta u_0 + scal_{g_H} \cdot u_0 = scal_{u_0^{4/n-2} \cdot g_H} \cdot u_0^{n+2/n-2}.$$

The new interesting outcome is that close to points $p \in \Sigma$ the deformed metric still looks like a cone: (C, \tilde{g}) is isometric to any of copy scaled around 0 and can be reparametrized as $c(\omega)^{4/n-2} \cdot g_{\mathbb{R}} + r^2 \cdot g_{\partial B_1(0) \cap C}$.

A delicate point is that this closeness depends *discontinuously* on the base point $p \in \Sigma$. But at least there is a radius for each p from that on (downwards) the balls around p belong to a well-controlled family (of subsets) of cone geometries. The fact that the ‘Hausdorff codimension’ of Σ is > 2 enters most visibly in the second stage of the deformation, which could be regarded as *stratified surgery* keeping $scal > 0$ along an *enhanced singular set*.

We start with (actually sums of inductively constructed bunches of) truncated standard Green’s functions (i.e. they look like Green’s functions in the interior of a ball (leaving $scal > 0$) and become constant near the boundary) defined on families of sufficiently small balls covering Σ .

The choice of radii is based on the accuracy of approximation H by tangent cones in a given point in Σ . We select a covering of Σ with upper bounded intersection number $c(n)$ (depending only on the dimension): this allows us to see that the sum of all negative contributions of the cut-off regions of our truncated Green’s functions is still compensated pointwise by the positive scalar curvature on H .

Now when we use such a sum to deform the metric close to Σ we also find that a suitable choice of coefficients leads to a barrier deflecting area minimizers within H homologically equivalent to a boundary of a neighborhood of Σ away from a small region around Σ .

Finally, using such a barrier and Theorem 2 we have a *smooth* $(n-2)$ -dimensional hypersurface N^{n-2} with positive mean curvature homologically equivalent to that boundary and arbitrarily close to Σ^{n-8} . Now a non-conformal deformation transforms a small one sided tube of N^{n-2} into a totally geodesic border (and additionally gives some extra $scal > 0$). Gluing this with a mirrored copy completes the argument.

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