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Calculus of Variations

A variational approach to a shape design problem for the wave equation

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Abstract

The problem of determining the optimal damping set for the stabilization of the wave equation may be not well-posed. By means of a vector variational reformulation and use of gradient Young measures, we present a general methodology to relax this kind of problems. From the optimal Young measure associated with the relaxed problem, we obtain information concerning minimizing sequences for the original problem as well as continuity properties of the relaxed cost function. ***To cite this article: A. Münch et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Relaxation d'un problème d'optimisation de forme pour l'équation des ondes. Le problème d'optimisation de la forme et de la position de la zone de dissipation pour l'équation des ondes peut être mal posé. En utilisant une reformulation variationnelle et la théorie de la mesure de Young, on présente dans cette note une méthode générale pour relaxer ce type de problème. A partir de la mesure de Young optimal associée au problème relaxé bien posé, on obtient des informations concernant les suites minimisantes pour le problème original ainsi que des propriétés de continuité sur la fonction coût relaxée. ***Pour citer cet article : A. Münch et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

On considère un domaine Ω borné et lipschitzien de $\mathbb{R}^N$, $N = 1, 2$, ω un sous ensemble de Ω de mesure de Lebesgue $|\omega|$ non nulle et a une fonction de $L^\infty(\Omega; \mathbb{R}^+)$. Pour tout temps $T > 0$ et toute donnée initiale $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, on note $u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ la solution de l'équation des ondes amortie (1) ($\mathcal{X}_\omega$ désigne la fonction caractéristique de ω) puis on considère le problème non linéaire (P) (voir (2)) qui consiste à déterminer la forme et la position optimale de $\omega \in \Omega_L = \{\omega \subset \Omega\colon\ |\omega| = L|\Omega|,\ 0 < L < 1\}$ minimisant l'intégrale sur $(0, T)$ de l'énergie associée à (1). Plusieurs travaux théoriques [1,2] et numériques [3,4] ont mis en évidence sur

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des problèmes similaires le caractère mal posé de ce type de problème (non existence d'un minimiseur dans la classe des fonctions caractéristiques). L'objet de cette Note est de ce fait de donner une formulation relaxée bien posée du problème (P). Précisément, le problème (RP) (introduit en (3)) où est solution de (4) est une relaxation de (P) : d'une part, (RP) admet un minimum dans $S_L = \{s \in L^\infty(\Omega;[0,1]), \int_\Omega s(x)\,dx = L|\Omega|\}$; par ailleurs, ce minimum coïncide avec l'infimum du problème (P). Enfin, la distribution de densité s est caractérisée par les laminés dits d'ordre 1 (voir Théorème 1.1). Il résulte que si la solution optimale s pour (RP) est telle que l'ensemble $\{x \in \Omega\colon\ 0 < s(x) < 1\}$ a une mesure positive, alors aucun sous-domaine ω possédant un nombre fini de composantes disjointes n'est optimal pour (P). Remarquons par ailleurs que le problème relaxé (RP) dérive de (P) en remplacant l'ensemble des fonctions caractéristiques de $L^\infty(\Omega;\{0,1\})$ par son enveloppe convexe dans $L^\infty(\Omega;[0,1])$.

Le problème relaxé s'obtient en quatre étapes (décrites en Section 2 pour $N = 1$) : la première consiste à reformuler (P) en un problème vectoriel (VP) (voir (9), (10)). La densité W qui apparaît étant non convexe, on la remplace par sa quasi-convexifiée CQW définie en (12) (voir [6]) ramenant le problème à la détermination des mesures définissant la classe $\mathcal{A}$ (voir (13)). Cette classe n'étant pas explicite, on procède comme dans [7] en l'élargissant à la classe $\mathcal{A}^\star$ des mesures poly-convexes, conduisant à la valeur exacte du polyconvexifié CPW (voir (20)), puis en montrant que cette valeur est en fait atteinte par les mesures appartenant à la classe $\mathcal{A}_\star$, dite des laminés d'ordre 1, elle-même incluse dans la classe $\mathcal{A}$ (de sorte que $\mathcal{A}_\star \subset \mathcal{A} \subset \mathcal{A}^\star$). Le Théorème 1.1 en résulte alors.

L'introduction de la formulation relaxée est justifiée par des simulations numériques. Pour de faibles valeurs de $a(x) = a\mathcal{X}_\Omega$, dépendant des données initiales, il apparaît que le problème (P) initial est bien posé et coïncide avec (RP) : la densité optimale unique est une fonction caractéristique. En revanche (P) devient mal posé dès que a excède une valeur critique : les densités optimales ne sont plus uniques et prennent des valeurs dans $]0,1[$. Le domaine ω optimal est composé d'un nombre arbitrairement grand d'intervals disjoints répartis sur Ω (voir [4,5]).

1. Introduction and statement of the result

Let us consider Ω, a bounded Lipschitz domain in $\mathbb{R}^N$, $N = 1, 2$, a subset ω of positive Lebesgue measure $|\omega|$, and a damping potential $a \in L^\infty(\Omega;\mathbb{R}^+)$. For any $T > 0$ and any $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ independent of ω and a, we denote by $u \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ the unique solution of the following damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)\mathcal{X}_\omega(x)u_t = 0 & \text{in } (0,T)\times\Omega,\\ u = 0 \quad \text{on } (0,T)\times\partial\Omega, & \big(u(0,\cdot),u_t(0,\cdot)\big) = (u_0,u_1) \quad \text{in } \Omega, \end{cases} \tag{1}$$

($\mathcal{X}_\omega$ designates the characteristic function of ω) and then consider the non-linear problem:

$$(P)\colon\quad \inf_{\omega\in\Omega_L} J(\mathcal{X}_\omega) \quad \text{where } J(\mathcal{X}_\omega) = \frac{1}{2}\int_0^T\int_\Omega \big(|u_t|^2 + |\nabla u|^2\big)\,dx\,dt \tag{2}$$

with $\Omega_L = \{\omega \subset \Omega\colon\ |\omega| = L|\Omega|,\ 0 < L < 1\}$. Problem (P) consists in finding the location and shape of ω which minimizes the integral of the energy over the time interval $(0,T)$. Several theoretical [1,2] and numerical [3,4] works for similar problems have exhibited the non-well posed character of this kind of problem (no existence of minimizer in the class of characteristic function). Therefore, this Note aims at giving a systematic variational justification of a well-posed relaxed reformulation of (P). Precisely, consider the optimization problem

$$(RP)\colon\quad \inf_{s\in S_L} \frac{1}{2}\int_0^T\int_\Omega \big(u_t^2 + |\nabla u|^2\big)\,dx\,dt \tag{3}$$

with $S_L = \{s \in L^\infty(\Omega;[0,1]), \int_\Omega s(x)\,dx = L|\Omega|\}$ and u (function of s) the unique solution of

$$\begin{cases} u_{tt} - \Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega,\\ u = 0 \quad \text{on } (0,T)\times\partial\Omega, & \big(u(0,\cdot),u_t(0,\cdot)\big) = (u_0,u_1) \quad \text{in } \Omega. \end{cases} \tag{4}$$

Then we have the following result:

Theorem 1.1. *Problem (RP) is a full relaxation of (P) in the sense that*

(i) *there are optimal solutions for (RP);*
(ii) *the infimum of (P) equals the minimum of (RP);*
(iii) *if s is optimal for (RP), and $\mathcal{X}_{\omega_j}$ converges weakly to s, then ω_j is a minimizing sequence for problem (P).*

Notice that (iii) above is equivalent to saying that the Young measure associated with an optimal sequence of damping sets is $s(x)\delta_1 + (1 - s(x))\delta_0$. This in turn implies that any sequence of damping sets in the form of a first-order laminate (in the 2-d case), with an arbitrary normal, which preserves the volume fraction given by the optimal s is a minimizing sequence for (P). This huge non-uniqueness of the distribution of the optimal damping sets may also be the explanation why the result of the numerical experiments depend on the initial densities (see [5]) though all of them have essentially the same cost. Notice that if the optimal solution s for (RP) is such that the subset $\{x \in \Omega\colon 0 < s(x) < 1\}$ has positive measure, then no finite admissible collection of subsets can be optimal for (P).

A second consequence of Theorem 1.1, part (iii), is that, because of the fact that the projection of the optimal Young measure onto the component corresponding to the field ∇u is trivial, the map

$$s \to J(s) = \frac{1}{2}\int_0^T\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) \mathrm{d}x\, \mathrm{d}t$$

is continuous for the weak* topology. This particular fact can also be obtained by more classical methods like Trotter–Kato's theorem (see [2]), although we believe our perspective is more general in scope.

2. Sketch of the proof of Theorem 1.1 in the case $N = 1$ ($\Omega = (0, 1)$)

We describe the proof here for the 1-d case. The proof of the 2-d case is more involved (though the final conclusion is the same), and can be found in [5]. The tools used include general techniques from non-convex, vector variational problems. As such, they can accommodate rather general situations, and are not restricted to specific ingredients of particular situations. We divide the proof in four main steps.

2.1. Reformulation of the shape problem as a constrained variational problem

The main idea here is to replace the non-local wave equation by a local (pointwise) constraint on a vector gradient. This requires the introduction of an additional auxiliary field. Namely, by taking advantage that ω is time independent, we rewrite the main equation in (1) as

$$\partial_t(u_t + a\mathcal{X}_\omega u) + \partial_x(-u_x) = \operatorname{div}(u_t + a\mathcal{X}_\omega u, -u_x) = 0, \quad \operatorname{div} \equiv (\partial_t, \partial_x). \tag{5}$$

Then, from the characterization of the 2-d free-divergence vector fields, there exists a scalar potential $v = v(t, x) \in H^1((0, T) \times \Omega)$ such that

$$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \bar{u} \quad \text{with } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

and $\nabla u = (u_t, u_x)^t$, $\nabla v = (v_t, v_x)^t$ and $\bar{u} = (u, 0)^t$. We introduce the vector field $U = (u, v) \in (H^1((0, T) \times \Omega))^2$ and the two sets of matrices

$$\Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2}\colon AM^{(1)} + BM^{(2)} = 0\right\}, \qquad \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2}\colon AM^{(1)} + BM^{(2)} = \lambda e_1\right\}, \tag{7}$$

where $M^{(i)}$, $i = 1, 2$, stands for the i-th row of the matrix M, $\lambda \in \mathbb{R}$ and $e_1 = (0, 1)^t$. Then introducing

$$W(x, U, M) = \begin{cases} \frac{1}{2}|M^{(1)}|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}}, \\ +\infty, & \text{otherwise}, \end{cases} \qquad V(x, U, M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}}, \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}}, \\ +\infty, & \text{otherwise} \end{cases} \tag{8}$$

problem (P) is equivalent to the following non-convex vector variational problem

$$(VP)\quad m \equiv \inf_{U\in(H^1((0,T)\times\Omega))^2} \int_0^T\int_\Omega W\big(x,U(t,x),\nabla U(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

subject to boundary, initial, and integral constraints

$$\begin{cases} U^{(1)}=0 \quad \text{on } (0,T)\times\partial\Omega, \quad \big(U^{(1)}(0,\cdot),U_t^{(1)}(0,\cdot)\big)=\big(u_0(\cdot),u_1(\cdot)\big) \quad \text{in } \Omega,\\ \displaystyle\int_0^1 V\big(x,U(t,x),\nabla U(t,x)\big)\,\mathrm{d}x = L \quad \text{in } (0,T). \end{cases} \tag{10}$$

2.2. Relaxation of the equivalent, variational reformulation

Since the integrand W is not convex, we cannot ensure the existence of solutions for (*VP*). We overcome this difficulty by considering a larger class of admissible solutions which includes all the minimizing sequences for (*VP*) and in a way that the resulting functional be weakly lower semi-continuous. This is a typical and general strategy for non-convex vector, variational problems. More precisely, let

$$\bar{m} = \inf_{U\in(H^1((0,T)\times\Omega))^2,\ s\in S_L} \left\{\int_0^T\int_\Omega CQW\big(t,x,\nabla U(t,x),s(x)\big)\,\mathrm{d}x\,\mathrm{d}t\right\} \tag{11}$$

where U satisfies the conditions (10-1). $CQW(t,x,\nabla U(t,x),s(x))$ stands for the constrained quasi-convexification of the density W and, for a fixed $(F,s)\in\mathcal{M}^{2\times 2}\times\mathbb{R}$, is defined as (see [6])

$$CQW(t,x,F,s) = \inf_\nu\left\{\int_{\mathcal{M}^{2\times2}} W(t,x,M)\,\mathrm{d}\nu(M):\ \nu\in\mathcal{A}(F,s)\right\}, \tag{12}$$

where

$$\mathcal{A}(F,s) = \left\{\nu:\ \nu \text{ is a homogeneous } H^1\text{-Young measure, } F=\int_{\mathcal{M}^{2\times2}} M\,\mathrm{d}\nu(M),\ \int_{\mathcal{M}^{2\times2}} V(M)\,\mathrm{d}\nu(M)=s\right\}. \tag{13}$$

Then, it can be proved (see [7]) that the infimum $\bar{m}$ is attained and that $m=\bar{m}$. However, we do not know explicitly the measures entering in the class $\mathcal{A}(F,s)$ that we need for computing $CQW(t,x,F,s)$.

2.3. Computation of the constrained quasiconvexification and identification of optimal Young measures

Following [7], in order to compute the integrand in (12), let us first minimize over the larger class of polyconvex measures $\mathcal{A}^*$, and then prove that the resulted infimum is actually attained at a measure which belongs to a class of measures, say $\mathcal{A}_*$, (the laminates) which is included in $\mathcal{A}(F,s)$, that is, $\mathcal{A}_*\subset\mathcal{A}\subset\mathcal{A}^*$. As a result, we will obtain the exact value of $CQW(t,x,F,s)$. The constrained polyconvexification, *CPW*, associated with the density W is given by

$$CPW(x,U,F,s) = \min_\nu \int_{\mathcal{M}^{2\times2}} W(x,U,M)\,\mathrm{d}\nu(M) \tag{14}$$

where the measure ν satisfies the constraints

$$\nu \text{ commutes with det}, \quad F=\int_{\mathcal{M}^{2\times2}} M\,\mathrm{d}\nu(M) \quad\text{and}\quad s=\int_{\mathcal{M}^{2\times2}} V(x,U,M)\,\mathrm{d}\nu(M). \tag{15}$$

From the volume constraint we deduce that this measure has the form $\nu=s\nu_1+(1-s)\nu_0$ with $\operatorname{supp}(\nu_j)\subset\Lambda_j$, $j=0,1$, and hence for each pair (F,s), the constrained polyconvexification $CPW(F,s)$ is computed by solving

$$\min_\nu \frac{s}{2}\int_{\Lambda_{1,\lambda}} \big|M^{(1)}\big|^2\,\mathrm{d}\nu_1(M) + \frac{1-s}{2}\int_{\Lambda_0}\big|M^{(1)}\big|^2\,\mathrm{d}\nu_0(M) \tag{16}$$

where ν is subject to

$$\nu \text{ commutes with det}, \quad F = s \int_{\Lambda_{1,\lambda}} M \, d\nu_1(M) + (1-s) \int_{\Lambda_0} M \, d\nu_0(M). \tag{17}$$

By introducing the variables $S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}$, $i = 1, 2$, where $\nu^{(1i)}$ stands for the projection of ν onto the $(1i)$-th component, and $F^j = \int_{\Lambda_j} M \, d\nu_j(M)$, $j = 0, 1$, and manipulating appropriately the restrictions associated with the linear manifolds Λ_j, and the important property of commutation of ν with the determinant leading to the following compatibility conditions

$$F_{12} = F_{21}, \qquad F_{11} = F_{22} + s\lambda, \tag{18}$$

we are led to consider a rather elementary linear programming problem. Indeed, we have to solve the problem

$$\text{Minimize in } \left(S_j, F_{11}^1\right): \quad \frac{1}{2}(S_1 + S_2) \tag{19}$$

subject to the conditions $S_1 - S_2 - s\lambda F_{11}^1 = \det F$ and $S_i \geqslant |F_{1i}|^2$, $i = 1, 2$. We obtain that the solution is $S_i = |F_{1i}|^2$, $i = 1, 2$, which implies that

$$CPW(F, s) = \frac{1}{2}\left|F^{(1)}\right|^2 \quad \text{if (18) holds} \quad \text{and} \quad +\infty \quad \text{otherwise}. \tag{20}$$

Notice now that the two equalities $S_i = |F_{1i}|^2$, $i = 1, 2$, imply, by the strict convexity of the 2-norm, that $\nu^{(1i)} = \delta_{F_{1i}}$, $i = 1, 2$. It is now straightforward to check that the unique, optimal ν furnishing the value of $CPW(F, s)$ is $\nu = (1-s)\delta_{G^0} + s\delta_{G^1}$, where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} + \lambda \end{pmatrix}. \tag{21}$$

Note that $G^j \in \Lambda_j$, $j = 0, 1$. Moreover, since $G^1 - G^0 = b \otimes n$, with $b = (0, \lambda)$ and $n = (0, 1)$, the optimal measure ν is a first-order laminate with normal n. As a matter of fact, this implies that the constrained quasi-convexification of the density W is also given by (20).

2.4. Reinterpretation in terms of the initial problem

To obtain Theorem 1.1, it suffices to reinterpret our result in terms of the initial variables. We put $\lambda = -a(x)U^{(1)}(t, x) = -a(x)u(t, x)$ which, in particular implies that the laminate depends on (t, x). However, since the direction of lamination is independent of time, we write $\nu = \{\nu_x\}_{x \in \Omega}$. We take into account the compatibility condition on $(0, T) \times \Omega$ of the first moment of the measure ν, namely, that the vector field $U \in (H^1((0, T) \times (0, 1)))^2$ satisfies

$$\nabla U(t, x) = \int_{\mathcal{M}^{2\times 2}} M \, d\nu_x(M) \quad \text{a.e. } (t, x) \in (0, T) \times \Omega \tag{22}$$

and the volume constraint $\int_{\mathcal{M}^{2\times 2}} V(x, U(t, x), \nabla U(t, x)) \, d\nu_x = s(x)$ with $s \in S_L$. Hence, the compatibility condition (18) reads as $u_x(t, x) = v_t(t, x)$ and $u_t(t, x) = v_x(t, x) - a(x)s(x)u(t, x)$ leading to (4-1) and the relaxed integrand of the cost function (20) as

$$CQW\left(x, U(t, x), \nabla U(t, x)\right) = \frac{1}{2}\left(u_t^2(t, x) + u_x^2(t, x)\right) \quad \text{if (4-1) holds}; \qquad +\infty \quad \text{otherwise}. \tag{23}$$

Finally, taking into account that the whole minimizing process is complete if we now minimize in $U(t, x)$ the expression

$$\int_0^T \int_0^1 CQW\left(x, U(t, x), \nabla U(t, x)\right) dx \, dt$$

and noticing that this is equivalent to minimizing in all possible functions $s \in S_L$ we arrive at problem (*RP*). The existence of optimal solutions for (*RP*) is a consequence of the fact that the density $CQW(x, U(t,x), \nabla U(t,x))$ is, by construction, quasi-convex. Moreover, the minima of (*RP*) coincide with the infima of (*VP*) because minimizers for (*RP*) are obtained in the form of a first-order laminate associated with a sequence of gradients of admissible vector fields $U^j(t,x)$ for (*VP*). This is the way in which the information concerning minimizing sequences for (*VP*) is codified in the relaxed problem (*RP*). That is, since the normal to the layers of all admissible laminates is $n = (0, 1)$, in the (t,x)-plane a minimizing sequence for (*VP*) looks like horizontal layers, limiting the regions of damping and in which for each time t the damping region is limited to have a total length of L.

3. Final remarks

In this Note, a new methodology based on the use of gradient Young measures as in [7] has been introduced for solving the non-linear optimization problem which consists in finding the optimal damping set for the stabilization of the wave equation. Precisely, a full relaxation of the original problem has been obtained by replacing the original cost function by its constrained quasi-convexification which is basically computed by solving the trivial mathematical programming problem (19). This relaxation is justified by some numerical experiments that indicate the non well-posed character of the original problem (see [5]).

This approach, which is typically non-linear, seems to be very appropriate to deal with some other more general optimization problems such as optimal design problems for the wave or heat equations where the designs appear in the principal part of the operator and/or with possibly more involved non-linear cost functions, optimization problems with non-linear state equations in divergence form, etc. It is of interest to find out how far we may go with this methodology.

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