

Complex Analysis

CR foliations, the strip-problem and Globevnik–Stout conjecture

Mark Agranovsky¹

Department of Mathematics and Statistics, Bar-Ilan University, Ramat-Gan, 52900, Israel

Received 30 January 2006; accepted 13 June 2006

Presented by Pierre Lelong

Abstract

We characterize CR functions on planar domains and real hypersurfaces in \mathbb{C}^2 in terms of analytic extendibility into attached analytic discs. It is done by studying propagation, from the boundary into interior, of degeneracy of CR foliations of solid torus-like manifolds. In particular, we answer, for smooth functions, two open questions mentioned in the title: about characterization of analytic functions in the complex plane and about characterization of boundary values of holomorphic functions in bounded domains in \mathbb{C}^n . *To cite this article: M. Agranovsky, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Des foliations CR, le problème ‘strip’ et la conjecture de Globevnik–Stout. On caractérise les fonctions CR sur les domaines dans le plan et sur les hypersurfaces dans \mathbb{C}^2 en termes de leur étendibilité aux disques analytiques attachés. Ça résulte de l’étude de la propagation, du bord à l’intérieur, de la dégénérescence des foliations CR des variétés de type tore solide. En particulier, pour les fonctions lisses on donne la réponse à deux questions ouvertes mentionnées dans le titre : sur la caractérisation des fonctions analytiques dans le plan complexe et sur la caractérisation des valeurs à bord des fonctions holomorphes dans un domaine borné. *Pour citer cet article : M. Agranovsky, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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1. Extensions of boundary foliations

Let n be an integer and M be a compact connected smooth oriented real $(2n - 1)$ -manifold with the boundary ∂M , possibly empty. Denote $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $\Sigma = \Delta \times M$, $b\Sigma = \partial\Delta \times M$, $\Sigma_0 = \Delta \times \partial M$.

By smooth family $\{D_t\}$ of analytic discs in \mathbb{C}^n , parametrized by the points $t \in M$, we understand the family of images $D_t = G(\Delta \times \{t\})$, where $G : \Sigma \rightarrow \mathbb{C}^n$ is smooth, analytic and diffeomorphic in $\zeta \in \Delta$. In the sequel, the parametrization G will be assumed real analytic in $\bar{\Sigma}$.

Definition. Let $d \leq m \leq 2n$ be integers. We say that a smooth mapping $G : \bar{\Sigma} \rightarrow \mathbb{C}^n$ is (m, d) -regular if (1) $\text{rank}_{\mathbb{R}} dG(p) = m$ for all $p \in \Sigma$, (2) $\text{rank}_{\mathbb{R}} dG|_{b\Sigma}(p) = d$ for all $p \in b\Sigma \setminus \text{Crit}(G)$, where $\text{Crit}(G) \subset b\Sigma$ is the $(d - 1)$ -dimensional critical manifold, contained in $G^{-1}(\partial\Omega)$, (3) $\text{rank}_{\mathbb{R}^n} dG|_{b\Sigma}(p) = d - 1$, for $p \in \text{Crit}(G)$.

E-mail address: agranovs@macs.biu.ac.il (M. Agranovsky).

¹ This work was partially supported by Israel Scientific Foundation, grant No. 279/02-01.

Under the above conditions, the analytic discs $D_t = G(\Delta \times \{t\})$ are attached by their boundaries to the real d -manifold $\Omega = G(b\Sigma)$ and cover this manifold.

Definition. We say that the family of analytic discs $\{D_t, t \in M\}$, and its parametrization G , are homologically non-trivial if $G(\{0\} \times M)$ represents a nonzero homology class in $H_k(G(\bar{\Sigma}), G(\bar{\Sigma}_0); \mathbb{R})$, $k = \dim M$.

We are interested in the two following cases of regular families of analytic discs:

Case A: $n = 1, m = 2, d = 2$. In this case $G(b\Sigma) = \Omega$ is a domain in \mathbb{C} , with the smooth boundary $\partial\Omega$ containing the critical values of G . The mapping $\zeta \mapsto G(\zeta, t)$ maps conformally the unit disc Δ onto the analytic disc $D_t \subset \mathbb{C}$. The restriction $G|_{b\Sigma}$ of G to the 2-dimensional boundary manifold $b\Sigma$ is a finitely sheeted covering over $\Omega \setminus \partial\Omega$. The sheets correspond to the curves $\gamma_t = \partial D_t$ passing through a fixed point in Ω .

Case B: $n = 2, m = 4, d = 3$. In this case Ω is a real 3-dimensional submanifold of \mathbb{C}^2 . The analytic discs $D_t = G(\Delta \times \{t\})$ are attached to Ω . The restriction of the parametrization mapping G to the 4-dimensional manifold $b\Sigma$ is a foliation with the 3-dimensional base $\Omega \setminus \partial\Omega$ and the 1-dimensional fibers $G^{-1}(b), b \in \Omega \setminus \partial\Omega$, corresponding to the curves ∂D_t passing through b .

The following theorem is the key one. It gives conditions of propagation of the degeneracy of CR mappings, from $b\Sigma$ to the manifold Σ :

Theorem 1. Let $Q = (F, G): \bar{\Sigma} \mapsto \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$ be a real-analytic CR-mapping, that is Q is holomorphic on each complex fiber $\Delta \times \{t\}$. Suppose that (i) G is $(2n, d)$ -regular; (ii) Q degenerates on the boundary $b\Sigma$, meaning that $F = f \circ G$ holds on $b\Sigma$ for some smooth function f on $\Omega = G(b\Sigma)$, (iii) G is homologically nontrivial. Suppose that either (A) $n = 1, d = 2$, or (B) $n = 2, d = 3$ and $H_1(M, \partial M) = 0$. Then Q degenerates in Σ , i.e., $F = \hat{f} \circ G$ on Σ , for some smooth function \hat{f} . The function f is CR-function in the interior of Ω .

The proof of Theorem 1 consists of two steps. A key point is a symmetry relation between mutual linking number associated with the mapping G and functions J on Σ , constant on the G -fibers $G^{-1}(z) \cap b\Sigma$. Following the ideas and the technique from [5] and [22], we compute the linking numbers by integration of Martinelli–Bochner differential forms and the needed relation follows from Stokes formula and the degeneracy of the mapping (J, G) on $b\Sigma$.

In the second, analytic, part of the proof we apply the symmetry relation to G and to the Jacobian J of the mapping $Q = (F, G)$, assuming that Q is nondegenerate and therefore the Jacobian is not identical zero. The contradiction with the assumption comes from comparing homology classes of the G -image of the critical set $J^{-1}(0)$ and of the J -image of the G -level sets. At this point we use the fact that the above critical set contains a nontrivial cycle in Σ which G -image is homologically nontrivial, by condition (iii). The homology classes are computed either by integration over cycles, in the case $\partial M = \emptyset$, or, if $\partial M \neq \emptyset$, by counting intersection indices of the relative cycles with transversal curves, using Poincaré duality.

The restriction $n \leq 2$ is needed for analyticity of J in ζ which guarantees that the G -images of J -zero cycles are co-oriented and no cancellation happens. We require real-analyticity to provide that the critical set of the mapping (F, G) is a cycle of proper dimension and, in particular, in order to have well defined integrals of differential forms over the critical set. This assumption likely can be reduced to differentiability by using technique of currents and more careful analysis of the zero sets of functions analytic in $\zeta \in \Delta$ and smooth in the parameter $t \in M$.

2. Applications of Theorem 1

2.1. The strip-problem

The following question was known since long ago:

Given a one-parameter family of Jordan curves in the plane and a (continuous or smooth) function having analytic extension inside any curve, decide when does this imply that the function is analytic on the union of the curves.

Earlier, this question was answered only for special families of the curves, mainly for circles [1,3,8–10,12,13,15,16,23–26], and for a very long time remained open for families of general type.

Theorem 1, case A, implies solution of the strip-problem in real analytic category for arbitrary generic families of Jordan curves without any restriction of geometric type. We prove that a generic family of Jordan curves does detect

analyticity if there is no point surrounded by all the curves. The importance of the latter condition was observed by Globevnik [10], on the example of circles. Our result is as follows:

Theorem 2. *Let D_t , $t \in M$, be a real-analytic (2, 2)-regular closed family of Jordan domains in \mathbb{C} , $M = S^1$ or $M = [0, 1]$. Assume (*) that the closed domains \overline{D}_t have no common point. Let f be a real-analytic function in $\overline{\Omega}$ satisfying the property: for each $t \in M$ the restriction $f|_{\partial D_t}$ admits holomorphic extension in D_t . Then f is holomorphic in the interior of $\Omega = \bigcup_{t \in M} \partial D_t$.*

2.2. Morera type theorem for hypersurfaces in \mathbb{C}^2

Theorem 1, case B, implies

Theorem 3. *Let $\mathcal{D} = \{D_t, t \in M\}$, be a real-analytic (4, 3)-regular family of analytic discs in \mathbb{C}^2 , attached by their boundaries to the real-analytic hypersurface $\Omega = \bigcup_{t \in M} \partial D_t$ and parametrized by a connected real 3-dimensional real-analytic compact manifold M . Suppose that $H_1(M, \partial M) = 0$ and (*) the family D_t , $t \in M$, is homologically nontrivial. If f is a real-analytic function on Ω and $f|_{\partial D_t}$ extends analytically in D_t for any $t \in M$ then f is CR function on Ω .*

The conditions (*) in Theorems 2 and 3 cannot be omitted. The condition (*) in Theorem 2 is equivalent to homological nontriviality of the family D_t .

2.3. Globevnik–Stout conjecture

The following conjecture generalizes the result for the complex ball due to Nagel and Rudin ([19], see also [20]) and was imposed in the article [17]:

Let D be a strictly convex bounded domain in \mathbb{C}^n and $S \subset D$ be a (smooth convex) closed hypersurface, compactly belonging to D . Suppose that a continuous (smooth) function f on the boundary ∂D has the property: for any point $t \in S$ and the complex line L_t , tangent to S at t , the restriction $f|_{L_t \cap \partial D}$ analytically extends to $L_t \cap D$. Then f is the boundary value of a holomorphic function in D .

Partial results were obtained in [2–4, 7, 11, 14, 18, 21]. Recently this conjecture was confirmed in [6] for geodesics in the Kobayashi metric in D in place of complex lines. Theorem 3 immediately implies Globevnik–Stout conjecture for the case when the surfaces ∂D and S and functions f are real analytic. Indeed, if $n = 2$ then the conjecture from [17] is a special case of Theorem 3 with $M = S$ and $D_t = D \cap L_t$, $t \in S$. If $n > 2$ then we apply Theorem 3 to intersections $D \cap \Pi$ with 2-dimensional complex planes Π and come up with the tangential Cauchy–Riemann equations for f on ∂D .

3. Reduction of Theorems 2 and 3 to Theorem 1

Let $G(\zeta, t)$ be a real-analytic regular parametrization of a family of analytic discs in Theorems 2 and 3. Let $F(\zeta, t)$ be the analytic extension of the function $\zeta \mapsto f(G(\zeta, t))$ from the unit circle $\{|\zeta| = 1\}$ to the unit disc $\{|\zeta| < 1\}$. Then the functions F and G satisfy the conditions of Theorem 1, when Theorem 2 corresponds to the case A and Theorem 3 – to the case B. The condition (iii) in Theorem 1 translates as the conditions (*) in Theorems 2 and 3 and the conclusions about f in Theorems 2 and 3 follow.

Theorem 2 states that the graph of f is a complex manifold in \mathbb{C}^2 . In this form, Theorem 1 generalizes to characterization of complex manifolds as real manifolds admitting nontrivial families of attached analytic discs:

Theorem 4. *Let $\Lambda \subset \mathbb{C}^2$ be a compact oriented real analytic manifold of real dimension 2, with $\partial \Lambda \neq \emptyset$. If Λ can be covered by boundaries of analytic discs, constituting a one-parameter real-analytic $(m, 2)$, $m \geq 2$, regular homologically nontrivial family, then Λ is a one-dimensional complex manifold in \mathbb{C}^2 , and Λ contains all the analytic discs.*

Acknowledgements

I am greatly indebted to Josip Globevnik for numerous stimulating and inspiring discussions. I thank Alexander Tumanov for informing me about his preprint [24].

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