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Probability Theory

Asymptotics for the distribution of lengths of excursions of a d-dimensional Bessel process ($0 < d < 2$)

Bernard Roynette [a], Pierre Vallois [a], Marc Yor [b,c]

[a] Université Henri-Poincaré, Institut Elie-Cartan, BP239, 54506 Vandoeuvre-les-Nancy cedex, France
[b] Laboratoire de probabilités et modèles aléatoires, Universités Paris VI et VII, 4, place Jussieu, case 188, 75252 Paris cedex 05, France
[c] Institut Universitaire de France, France

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Presented by Marc Yor

Abstract

Let $(R_t, t \geqslant 0)$ denote a d-dimensional Bessel process ($0 < d < 2$). For every $t \geqslant 0$, we consider the times $g_t = \sup\{s \leqslant t\colon\ R_s = 0\}$, and $d_t = \inf\{s > t\colon\ R_s = 0\}$, as well as the three sequences: $(V^n_{g_t}, n \geqslant 1)$, $(V^n_t, n \geqslant 2)$, and $(V^n_{d_t}, n \geqslant 2)$, which consist of the lengths of excursions of R away from 0 before g_t, before t, and before d_t, respectively, each one being ranked by decreasing order.

We obtain a limit theorem concerning each of the laws of these three sequences, as $t \to \infty$. The result is expressed in terms of a positive, σ-finite measure Π on the set $\mathcal{S}^{\downarrow}$ of decreasing sequences. Π is closely related with the Poisson–Dirichlet laws on $\mathcal{S}^{\downarrow}$. ***To cite this article: B. Roynette et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Asymptotiques pour la distribution des longueurs des excursions d'un processus de Bessel de dimension d ($0 < d < 2$). Soit $(R_t, t \geqslant 0)$ un processus de Bessel de dimension $d \in (0, 2)$. Pour tout $t \geqslant 0$, on considère les temps $g_t = \sup\{s \leqslant t\colon\ R_s = 0\}$ et $d_t = \inf\{s > t\colon\ R_s = 0\}$, ainsi que les trois suites : $(V^n_{g_t}, n \geqslant 1)$, resp. $(V^n_t, n \geqslant 2)$, resp. $(V^n_{d_t}, n \geqslant 2)$ des longueurs d'excursions de R hors de 0, avant g_t, resp. avant t, resp. avant d_t, rangées par ordre décroissant.

Nous obtenons un théorème limite concernant chacune des lois de ces trois suites, lorsque $t \to \infty$. Ce théorème s'exprime à l'aide d'une mesure positive, σ-finie, Π sur $\mathcal{S}^{\downarrow} = \{s = (s_1, s_2, \ldots, s_n, \ldots);\ s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant \cdots \geqslant 0\}$. Π est intimement liée aux lois de Poisson–Dirichlet sur $\mathcal{S}^{\downarrow}$. ***Pour citer cet article : B. Roynette et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Soit $(R_t, t \geqslant 0; P^{(\alpha)})$ un processus de Bessel de dimension $d = 2(1-\alpha)$ ($0 < \alpha < 1$, ou $0 < d < 2$). Pour tout t, soit $V^1_t \geqslant V^2_t \geqslant V^3_t \geqslant \cdots$ la suite des longueurs des excursions hors de 0 du processus R pendant l'intervalle de temps

E-mail addresses: Bernard.Roynette@iecn.u-nancy.fr (B. Roynette), Pierre.Vallois@iecn.u-nancy.fr (P. Vallois), deaproba@proba.jussieu.fr (M. Yor).

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$[0,t]$, classées par ordre décroissant. Dans cette liste, $(V_t^1,\dots,V_t^n,\dots)$, figure l'âge $(t-g_t)$ de l'excursion débutant en $g_t=\sup\{s\leqslant t\colon\ R_s=0\}$ et finissant en $d_t=\inf\{s>t\colon\ R_s=0\}$.

Nous nous intéressons, dans cette Note, à l'étude asymptotique, quand $t\to\infty$, des lois des 3 suites $(V_{g_t}^i; i=1,2,\dots)$, $(V_t^i; i=2,3,\dots)$ et $(V_{d_t}^i; i=2,3,\dots)$. Nos principaux résultats sont présentés dans les Théorèmes 0.1 et 0.2 ci-dessous.

Asymptotique de la loi de $(V_{g_t}^1, V_{g_t}^2,\dots)$

On note $W^i (i=1,2,\dots)$ la suite des coordonnées de l'ensemble $\mathcal{S}^\downarrow=\{s=(s_1,s_2,\dots,s_n,\dots); s_1\geqslant s_2\geqslant s_3\geqslant\cdots\}$, c'est-à-dire : $W^i(s)=s_i$.

Théorème 0.1. (i) *Il existe une mesure positive, σ-finie, Π sur $\mathcal{S}^\downarrow$ ne dépendant pas de $\alpha\in(0,1)$, telle que, pour toute $F:\mathcal{S}^\downarrow\to\mathbb{R}_+$, borélienne, bornée, à support compact en la première variable, on ait* :

$$\begin{aligned} E_\Pi\big[F\big(W^1,W^2,\dots\big)\big] &:= \lim_{t\to\infty} t^\alpha E^{(\alpha)}\big[F\big(\big(V_{g_t}^i\big)^\alpha; i=1,2,\dots\big)\big] \\ &= \int_0^\infty E\bigg[F\bigg(x, x\frac{T_2}{T_3}, x\frac{T_2}{T_4},\dots, x\frac{T_2}{T_{n+1}},\dots\bigg)\bigg]\,\mathrm{d}x \\ &= \int_0^\infty E\big[F(x, x\rho_2, x\rho_2\rho_3,\dots, x\rho_2\cdots\rho_n,\dots)\big]\,\mathrm{d}x \end{aligned}$$

où, dans les deux dernières expressions :

- $\rho_2,\rho_3,\dots,\rho_n,\dots$ *sont des v.a. indépendantes, et pour tout n, $\rho_n \overset{(loi)}{=} U^{1/n}$ suit la loi beta de paramètre $(n,1)$ et U est uniforme sur $(0,1)$* ;
- $(T_i, i\geqslant 1)$ *est la suite croissante des instants de sauts d'un processus de Poisson standard.*

(ii) *Sous la mesure Π, la suite (W^n) possède les propriétés suivantes* :

- *W^1 est distribuée selon la mesure de Lebesgue sur $\mathbb{R}_+$, et est indépendante de*

$$\bigg(\frac{W^2}{W^1},\frac{W^3}{W^1},\dots,\frac{W^n}{W^1},\dots\bigg) \overset{(loi)}{=} \bigg(\frac{T_2}{T_3},\frac{T_2}{T_4},\dots,\frac{T_2}{T_{n+1}},\dots\bigg).$$

- *Plus généralement, pour tout n, W^n est distribuée selon n fois la mesure de Lebesgue sur $\mathbb{R}_+$, et est indépendante de*

$$\bigg(\frac{W^{n+1}}{W^n},\dots,\frac{W^{n+k}}{W^n},\dots\bigg) \overset{(loi)}{=} \bigg(\frac{T_{n+1}}{T_{n+2}},\frac{T_{n+1}}{T_{n+3}},\dots,\frac{T_{n+1}}{T_{n+k+1}},\dots\bigg).$$

Asymptotique des lois des suites $(V_t^i; i\geqslant 2)$ et $(V_{d_t}^i; i\geqslant 2)$

Théorème 0.2. *Soit F comme dans le Théorème* 0.1.

Les quantités $t^\alpha E^{(\alpha)}[F(V_t^i; i\geqslant 2)]$ et $t^\alpha E^{(\alpha)}[F(V_{d_t}^i; i\geqslant 2)]$ convergent toutes deux, lorsque $t\to\infty$, vers la même limite, qui est égale à :

$$E_\Pi\big[F\big(W^1,W^2,\dots\big)\big].$$

On notera que les suites $(V_t^i; i\geqslant 2)$ et $(V_{d_t}^i; i\geqslant 2)$ sont « décalées d'un indice », et ne font figurer ni V_t^1, ni $V_{d_t}^1$. L'explication intuitive de ce fait est donnée par le Théorème 6 ci-dessous, et résulte du Lemme 5.

1. The Poisson–Dirichlet distributions $P_{\alpha,0}$ and $P_{\alpha,\alpha}$

1) Throughout this Note, we consider $(R_t, t \geqslant 0; P^{(\alpha)})$ a Bessel process of dimension d, with $0 < d < 2$, $d = 2(1-\alpha)$, started from 0. Let $V_t^1 \geqslant V_t^2 \geqslant V_t^3 \geqslant \cdots$ denote the sequence of lengths of its excursions away from 0, during the time interval $[0,t]$, ranked in decreasing order. In particular, the so-called age $(t - g_t)$, where $g_t = \sup\{s \leqslant t{:}\ R_s = 0\}$, is included in the sequence $(V_t^1, \ldots, V_t^n, \ldots)$.

2) By scaling, the law of:

$$\frac{1}{t}\left(V_t^1, V_t^2, \ldots\right) \tag{1}$$

does not depend on t, it has some remarkable properties, and is found naturally in a number of probabilistic studies. Its distribution, the Poisson–Dirichlet distribution with parameter $(\alpha, 0)$, is denoted by $P_{\alpha,0}$ in [6].

$P_{\alpha,0}$ is also (see [5]) the distribution of:

$$\frac{1}{\tau_\ell}\left(V_{\tau_\ell}^1, \ldots, V_{\tau_\ell}^n, \ldots\right) \tag{2}$$

where $(\tau_\ell, \ell \geqslant 0)$ denotes any (choice of) inverse local time, i.e.:

$$\tau_\ell = \inf\{t \geqslant 0{:}\ L_t > \ell\} \tag{3}$$

for $(L_t, t \geqslant 0)$ a local time at 0 for $BES(d)$ (see [1] for a discussion of the different choices found in the literature). The identity in law between (1) and (2) was obtained in [5] – see also [7] where other random times σ than τ_ℓ are exhibited, such that the sequence (2) with σ instead of τ_ℓ still has the same distribution – these results may be considered as a reinforcement of Lévy's result for the time spent in $\mathbb{R}_+$ by a Brownian motion $(B_t, t \geqslant 0)$:

$$\text{if } A_t^+ := \int_0^t 1_{(B_s > 0)}\,\mathrm{d}s,\ \text{then both } \frac{1}{t}A_t^+ \text{ and } \frac{1}{\tau_\ell}A_{\tau_\ell}^+ \text{ are arc-sine distributed.}$$

3) Likewise, if in (1), we replace t by

$$g_t := \sup\{s \leqslant t{:}\ R_s = 0\} \tag{4}$$

then, the variable:

$$\frac{1}{g_t}\left(V_{g_t}^1, \ldots, V_{g_t}^n, \ldots\right) \tag{5}$$

is independent from g_t, and is distributed as:

$$\left(v^1, v^2, \ldots, v^n, \ldots\right) \tag{6}$$

the sequence of ranked excursions of the standard $BES(d)$ bridge; its distribution, the Poisson–Dirichlet distribution with parameter (α, α) is denoted by $P_{\alpha,\alpha}$ in [6].

4) Here is a description of $P_{\alpha,0}$ and $P_{\alpha,\alpha}$ on the canonical space $\mathcal{S}^{\downarrow}$ of decreasing sequences $s = \{s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant \cdots \geqslant 0\}$, where we denote $W^i(s) = s_i$ the sequence of coordinates.

Theorem 1. *([6]) Let $\{T_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i, i = 1, 2, \ldots\}$ denote the sequence of jump times of a standard Poisson process with parameter 1, i.e.: the ε_i are i.i.d. standard exponential variables. Then:*

(i) *Under $P_{\alpha,0}$, the sequence $(W^i = (V^i)^\alpha; i = 1, 2, \ldots)$ is distributed as*

$$\left(\left\{T_i\left(\sum_{m=1}^{\infty} T_m^{-1/\alpha}\right)^\alpha\right\}^{-1}; i = 1, 2, \ldots\right). \tag{7}$$

Consequently, $L := \lim_{n\to\infty} nW^n$ exists $P_{\alpha,0}$ a.s., and is distributed (jointly with the W^i's) as

$\left(\sum_{m=1}^{\infty} T_m^{-1/\alpha}\right)^{-\alpha}$. *In other terms*:

$$\left(L, \frac{W^i}{L}; i = 1, 2, \ldots\right) \overset{(law)}{=} \left(\left(\sum_{m=1}^{\infty} T_m^{-1/\alpha}\right)^{-\alpha}, \frac{1}{T_i}; i = 1, 2, \ldots\right). \tag{8}$$

(ii) *A closely related and useful description of $P_{\alpha,0}$ is*:

$$\begin{aligned}\left(W^2, W^3, \ldots, W^{n+1}, \ldots\right) &\overset{(law)}{=} \left(W^1 \frac{T_1}{T_2}, W^1 \frac{T_1}{T_3}, \ldots, W^1 \frac{T_1}{T_{n+1}}, \ldots\right) \\ &\overset{(law)}{=} \left(W^1 \rho_1, W^1 \rho_1 \rho_2, \ldots, W^1 \rho_1 \rho_2 \cdots \rho_n, \ldots\right)\end{aligned} \tag{9}$$

where $(\rho_n = \frac{T_n}{T_{n+1}}, n \geqslant 1)$ *is a sequence of independent variables, and* ρ_n *is* beta$(n, 1)$ *distributed. Note that, from* (9), *one has*:

$$\left(W^1\right)^{1/\alpha} + \sum_{n=1}^{\infty} \left(W^1\right)^{1/\alpha} (\rho_1 \cdots \rho_n)^{1/\alpha} = 1 \tag{10}$$

so that W^1 is determined from the ρ_i's.

(iii) *$P_{\alpha,\alpha}$ is absolutely continuous with respect to $P_{\alpha,0}$, with*:

$$P_{\alpha,\alpha} = \mathcal{C}_\alpha \, L \bullet P_{\alpha,0} \tag{11}$$

where:

$$\mathcal{C}_\alpha = B(1+\alpha, 1-\alpha) = \Gamma(1+\alpha)\Gamma(1-\alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)}. \tag{12}$$

2. Asymptotic distribution of $(V_{g_t}^1, V_{g_t}^2, \ldots)$

1) Motivated by the study of the existence and description of the penalised laws:

$$\lim_{t\to\infty} \frac{1_{(V_{g_t}^i \leqslant x_i; i \leqslant k)} \bullet P^{(\alpha)}}{P^{(\alpha)}(V_{g_t}^i \leqslant x_i; i \leqslant k)} \tag{13}$$

where $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$ is a fixed, finite, decreasing sequence of positive reals (see [8,9]), we studied the asymptotics of the denominator of (13), and obtained the following:

Theorem 2. (i) *There exists a positive, σ-finite measure Π on $\mathcal{S}^\downarrow$, which does not depend on $\alpha \in (0, 1)$, such that, for every $F : \mathcal{S}^\downarrow \to \mathbb{R}_+$, bounded, Borel, and with compact support in the first variable*:

$$\begin{aligned}E_\Pi\left[F\left(W^1, W^2, \ldots\right)\right] &:= \lim_{t\to\infty} t^\alpha E^{(\alpha)}\left[F\left(\left(V_{g_t}^i\right)^\alpha; i = 1, 2, \ldots\right)\right] \\ &= \int_0^\infty E\left[F\left(x, x\frac{T_2}{T_3}, x\frac{T_2}{T_4}, \ldots, x\frac{T_2}{T_{n+1}}, \ldots\right)\right] \mathrm{d}x \\ &= \int_0^\infty \mathrm{d}x \, E\left[F(x, x\rho_2, x\rho_2\rho_3, \ldots, x\rho_2 \cdots \rho_n, \ldots)\right]\end{aligned} \tag{14}$$

with the same notation as in Theorem 1.

(ii) *The measure Π enjoys the following properties*:

(a) *Under Π, W^1 is distributed as Lebesgue measure on $\mathbb{R}_+$, and is independent from*:

$$\left(\frac{W^2}{W^1}, \ldots, \frac{W^k}{W^1}, \ldots\right) \overset{(law)}{=} \left(\frac{T_2}{T_3}, \ldots, \frac{T_2}{T_{k+1}}, \ldots\right). \tag{15}$$

(b) *More generally, under Π, for any n, W^n is distributed as n times Lebesgue's measure on $\mathbb{R}_+$ and is independent of*:

$$\left(\frac{W^{n+1}}{W^n},\ldots,\frac{W^{n+k}}{W^n},\ldots\right)\overset{(law)}{=}\left(\frac{T_{n+1}}{T_{n+2}},\ldots,\frac{T_{n+1}}{T_{n+k+1}},\ldots\right)$$
$$\overset{(law)}{=}(\rho_{n+1},\rho_{n+1}\rho_{n+2},\ldots,\rho_{n+1}\cdots\rho_{n+k},\ldots). \tag{16}$$

(c) *Under Π, the density of $(W^1,\ldots,W^n)$ is*:

$$f_n(s_1,\ldots,s_n)=\frac{(n!)(s_n)^{n-1}}{(s_1s_2\cdots s_{n-1})^2}1_{(0\leqslant s_n\leqslant\cdots\leqslant s_1)}. \tag{17}$$

(d) *Shifting the sequence $(W^k,k\geqslant 1)$ into $(W^{n+k},k\geqslant 1)$, for any given $n\geqslant 1$, has the following effect on Π*:

$$E_\Pi\big[h\big(W^n,\ldots,W^{n+p}\big)\big]=\binom{n+p}{p+1}E_\Pi\left[\left(\frac{W^{p+1}}{W^1}\right)^{n-1}h\big(W^1,\ldots,W^{p+1}\big)\right] \tag{18}$$

for any $h:\mathbb{R}_+^{p+1}\to\mathbb{R}_+$, Borel.

We have obtained two proofs of Theorem 2, and we now present their main ingredients:

(i) Our first proof relies on the description of $P_{\alpha,\alpha}$ stated in the above Theorem 1, and upon the elementary remark: since g_t/t is beta$(\alpha,1-\alpha)$ distributed, then:

$$t^\alpha E\big[\varphi\big((g_t)^\alpha\big)\big]\underset{t\to\infty}{\longrightarrow}\frac{1}{B(1+\alpha,1-\alpha)}\int_0^\infty \mathrm{d}x\,\varphi(x) \tag{19}$$

for any $\varphi:\mathbb{R}_+\to\mathbb{R}_+$, Borel, with compact support.

(ii) Our second proof does not depend on Theorem 1; instead, it relies upon the following:

Lemma 3. *Let $n\geqslant 1$, and $y_1\geqslant y_2\geqslant\cdots\geqslant y_n\geqslant 0$, a decreasing sequence of n real numbers. Then (all functions introduced below do not depend on α)*:

(i) *There exist functions P_n of n variables such that*:

$$P^{(\alpha)}\big[\big(V_{g_t}^1\big)^\alpha\leqslant y_1,\ldots,\big(V_{g_t}^n\big)^\alpha\leqslant y_n\big]\underset{t\to\infty}{\sim}\frac{1}{t^\alpha}P_n(y_1,\ldots,y_n). \tag{20}$$

(ii) *For any $p\geqslant 2$, there exists a polynomial Q_p in p variables, which is homogeneous of degree $(p-2)$, such that:*

$$P_n(y_1,\ldots,y_n)=y_n+\sum_{p=2}^n y_n^p\left(\frac{1}{y_n}-\frac{1}{y_{p-1}}\right)Q_p\left(\frac{1}{y_1},\frac{1}{y_2},\ldots,\frac{1}{y_{p-1}},\frac{1}{y_n}\right). \tag{21}$$

In particular, $Q_2\equiv 1$, $Q_3(x_1,x_2,x_3)=x_3+x_2-2x_1$, $Q_4(x_1,x_2,x_3,x_4)=x_3^2+x_3x_4+x_4^2-3x_3x_1-3x_2^2-3x_1x_4+6x_1x_2$.

(iii) *The polynomials $(Q_n,n\geqslant 2)$ satisfy the following recurrence relation*:

$$Q_n(x_1,\ldots,x_n)=\sum_{p=1}^{n-2}\binom{n-1}{p}(x_n-x_{n-1})^{p-1}(x_{n-1}-x_{n-p-1})$$
$$\cdots Q_{n-p}(x_1,x_2,\ldots,x_{n-p-2},x_{n-p-1},x_{n-1})+(x_n-x_{n-1})^{n-2}. \tag{22}$$

(iv) *The coefficient of $x_1x_2\cdots x_{n-2}$ in $Q_n(x_1,\ldots,x_n)$ is equal to $(-1)^n(n-1)!$*

(v) *The nth mixed partial derivative of P_n, with respect to all variables is*:

$$\frac{\partial^n P_n}{\partial y_1\cdots\partial y_n}(y_1,\ldots,y_n)=\frac{n!\,y_n^{n-1}}{(y_1y_2\cdots y_{n-1})^2} \tag{23}$$

for $y_n<y_{n-1}<\cdots<y_1$.

Our second proof of Theorem 2 easily follows from Lemma 3: in particular, with the help of (20) and (23), we obtain, that for any $F : \mathcal{S}_n^{\downarrow} \to \mathbb{R}_+$, bounded, and with compact support in the first variable:

$$t^\alpha E^{(\alpha)}\big[F\big(\big(V_{g_t}^1\big)^\alpha, \ldots, \big(V_{g_t}^n\big)^\alpha\big)\big] \text{ converges towards:}$$

$$\int_{\mathcal{S}_n^{\downarrow}} F(s_1, \ldots, s_n) \frac{n! s_n^{n-1}}{(s_1 s_2 \cdots s_{n-1})^2} \, ds_1 \cdots ds_n \equiv \int_{\mathcal{S}^{\downarrow}} F(s_1, \ldots, s_n) \, d\Pi(s).$$

3. Asymptotic distributions of $(V_t^2, V_t^3, \ldots)$ and $(V_{d_t}^2, V_{d_t}^3, \ldots)$

The next theorem is a companion to Theorem 2.

Theorem 4. *With the same hypotheses concerning F as in Theorem 2, both quantities*:

$$\lim_{t\to\infty} t^\alpha E^{(\alpha)}\big[F\big(\big(V_t^2\big)^\alpha, \ldots, \big(V_t^{n+1}\big)^\alpha, \ldots\big)\big] \quad \textit{and} \tag{24}$$

$$\lim_{t\to\infty} t^\alpha E^{(\alpha)}\big[F\big(\big(V_{d_t}^2\big)^\alpha, \ldots, \big(V_{d_t}^{n+1}\big)^\alpha, \ldots\big)\big] \tag{25}$$

$$\textit{exist and are equal to}: E_\Pi\big[F\big(W^1, W^2, \ldots, W^n, \ldots\big)\big]. \tag{26}$$

Note the shift of indices between the expressions found in (24) and (25) on one hand, and (26) on the other hand. Intuitively, the explanation of the absence of V_t^1 in (24) comes from the fact that asymptotically, V_t^1 is much bigger than the other excursion lengths (see Theorem 6 below for a precise assertion of this fact; see [3] for the law of V_t^1), and that, with a large probability, $V_t^1 = t - g_t$. Indeed, from [5]: $P(t - g_t = V_t^i) = E[\frac{V_t^i}{t}]$.

Hence, this sequence in i is decreasing. More completely, we obtained the following asymptotics:

Lemma 5. *For any $n \geqslant 2$, and $x_1 > x_2 > \cdots > x_n$*:

$$\text{(i) } P^{(\alpha)}\big(V_t^2 \leqslant x_1, \ldots, V_t^{n+1} \leqslant x_n, t - g_t = V_t^1\big) = P^{(\alpha)}\big(V_{g_t}^1 \leqslant x_1, \ldots, V_{g_t}^n \leqslant x_n, t - g_t > V_{g_t}^1\big)$$
$$\underset{t\to\infty}{\sim} P^{(\alpha)}\big(V_{g_t}^1 \leqslant x_1, \ldots, V_{g_t}^n \leqslant x_n\big), \tag{27}$$

$$\text{(ii) } P^{(\alpha)}\big(V_t^2 \leqslant x_1, \ldots, V_t^{n+1} \leqslant x_n, t - g_t \leqslant V_t^2\big) = \mathrm{o}\big(t^{-\alpha}\big). \tag{28}$$

Lemma 5 still holds as we replace in (27) and (28) the sequence $(V_t^2, \ldots, V_t^{n+1})$ by $(V_{d_t}^2, \ldots, V_{d_t}^{n+1})$.

4. On one-dimensional asymptotics

1) Earlier in our study, we had obtained only one-dimensional asymptotics, that is estimates as $t \to \infty$, for: $P^\alpha((V_{g_t}^n)^\alpha \leqslant x)$, and similarly with V_t^n, and $V_{d_t}^n$.

We now give some details about the different tools we used to obtain these estimates.

2) We denote by $\Phi(\alpha, \gamma; \bullet)$ $(\gamma \neq 0, -1, -2)$ the confluent hypergeometric function of index (α, γ):

$$\Phi(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!} \quad (z \in \mathbb{C}) \tag{29}$$

with $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ (cf. [4], p. 260).

Let $\lambda_0(\alpha)$ denote the first positive zero of:

$$\theta_\alpha(\lambda) = 1 - \lambda\, e^{-\lambda} \frac{1}{1-\alpha} \Phi(1-\alpha, 2-\alpha; \lambda). \tag{30}$$

We are now able to present our one-dimensional estimates for $n = 1$.

Theorem 6. *The three asymptotic results hold*:

(i) $$t^\alpha P^{(\alpha)}\big(V^1_{g_t} \leqslant x\big) \underset{t\to\infty}{\longrightarrow} x^\alpha. \tag{31}$$

(ii) *There exists a constant* $\mathcal{C}(\alpha)$ *such that*:

$$P^{(\alpha)}\big(V^1_t \leqslant x\big) \underset{t\to\infty}{\sim} \mathcal{C}(\alpha)\exp\left(-\lambda_0(\alpha)\frac{x}{t}\right) \quad \text{and} \tag{32}$$

$$P^{(\alpha)}\big(V^1_{d_t} \leqslant x\big) \underset{t\to\infty}{\sim} \big(\mathcal{C}(\alpha)\exp\big(-\lambda_0(\alpha)\big)\big)\exp\left(-\lambda_0(\alpha)\frac{x}{t}\right). \tag{33}$$

Our main tool for the proof of Theorem 6 is

Lemma 7. *Let* $x \geqslant 0$*, and* $H^{(1)}_x = \inf\{t \geqslant 0\colon\ t - g_t > x\}$. *Then, there exist two functions* $\psi, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*:

$$\text{(i) } \psi(x\beta) := E^{(\alpha)}\big(\exp\big(-\beta H^{(1)}_x\big)\big) = \frac{1}{\Phi(1, 1-\alpha; \beta x)} \quad \text{and}$$

$$\Phi(1, 1-\alpha; \beta x) = 1 + (\beta x)\,\mathrm{e}^{\beta x}\,\frac{1}{1-\alpha}\Phi(1-\alpha, 2-\alpha; -\beta x). \tag{34}$$

The function $\beta \to \psi(\beta)$ *is holomorphic in the strip*: $\operatorname{Re}\beta > -\lambda_0(\alpha)$. *In particular*:

$$\psi(x\beta) = 1 - \beta x\, E^{(\alpha)}\big(H^{(1)}_1\big) + \mathrm{o}(\beta) = 1 - \frac{\beta x}{1-\alpha} + \mathrm{o}(\beta) \qquad (\beta \to 0) \tag{35}$$

$$\text{(ii) } \theta(x\beta) := E^{(\alpha)}\big[E^{(\alpha)}_{R_{H^{(1)}_x}}\big(\exp(-\beta T_0)\big)\big] = \Phi(1, 1-\alpha; \beta x) - \Gamma(1-\alpha)(\beta x)^\alpha \exp(\beta x) \tag{36}$$

$$= \Gamma(1-\alpha)(\beta x)^\alpha + \mathrm{o}\big(\beta^\alpha\big) \quad (\beta \to 0) \tag{37}$$

where $T_0 := \inf\{t\colon\ R_t = 0\}$ *and* $P^{(\alpha)}_r$ *denotes the law of the BES*(α) *process starting from* r *at time* $t = 0$.

Note that formula (36) had already been obtained under a slightly different form in ([6], Proposition 11).

We now prove point (i) of Theorem 6:

Let $\mathcal{S}_\beta$ be an exponential variable with parameter β, independent from $(R_t, t \geqslant 0)$. We have:

$$\begin{aligned}
P^{(\alpha)}\big(V^1_{g_{\mathcal{S}_\beta}} \leqslant x\big) &= P^{(\alpha)}(\mathcal{S}_\beta \leqslant d_{H^{(1)}_x}) = 1 - E^{(\alpha)}\big(\exp(-\beta d_{H^{(1)}_x})\big)\\
&= 1 - E^{(\alpha)}\big(\exp\big(-\big(\beta H^{(1)}_x + T_0 \circ \Theta_{H^{(1)}_x}\big)\big)\big)\\
&= 1 - E^{(\alpha)}\big(\exp\big(-\beta H^{(1)}_x\big)\big)E^{(\alpha)}\big(E^{(\alpha)}_{R_{H^{(1)}_x}}\big(e^{-\beta T_0}\big)\big)\\
&\qquad \text{(from the independence of } H^{(1)}_x \text{ and } R_{H^{(1)}_x} \text{ (see, e.g. [2]))}\\
&= 1 - \psi(x\beta)\theta(x\beta).
\end{aligned}$$

On the other hand,

$$P^{(\alpha)}\big(V^1_{g_{\mathcal{S}_\beta}} \leqslant x\big) = \beta\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\beta t} P^{(\alpha)}\big(V^1_{g_t} \leqslant x\big).$$

Hence:

$$\int_0^\infty \mathrm{e}^{-\beta t} P^{(\alpha)}\big(V^1_{g_t} \leqslant x\big)\,\mathrm{d}t \underset{\beta\to 0}{\sim} \Gamma(1-\alpha)x^\alpha\beta^{\alpha-1}$$

and we conclude with the help of the Tauberian theorem.

The proofs of points (ii) and (iii) of Theorem 6 follow in the same manner, although we then use the Mellin–Fourier transform instead of the Tauberian theorem (and (34)).

3) Here are now our one-dimensional results for $n \geqslant 2$.

Theorem 8. *The three asymptotic results hold*:

(i) *For any* $n \geqslant 1$, $\lim_{t\to\infty} t^\alpha P^{(\alpha)}\big(V_{g_t}^n \leqslant x\big) = nx^\alpha$. (38)

(ii) *For any* $n \geqslant 2$, $\lim_{t\to\infty} t^\alpha P^{(\alpha)}\big(V_t^n \leqslant x\big) = (n-1)x^\alpha$. (39)

(iii) *For any* $n \geqslant 2$, $\lim_{t\to\infty} t^\alpha P^{(\alpha)}\big(V_{d_t}^n \leqslant x\big) = (n-1)x^\alpha$. (40)

The proof of Theorem 8 is quite similar to that of Theorem 6. It hinges upon the following:

Lemma 9. *Define, by iteration, for any* $n \geqslant 1$, *the sequence of stopping times*:

$$H_x^{(n+1)} = d_{H_x^{(n)}} + H_x^{(1)} \circ \Theta_{d_{H_x^{(n)}}} \tag{41}$$

where $(\Theta_u, u \geqslant 0)$ *are the usual time-shift operators. Then*:

(i) $\psi_n(\beta x) := E^{(\alpha)}\big(\exp\big(-\beta H_x^{(n)}\big)\big) = \big(\psi(\beta x)\big)^n \big(\theta(\beta x)\big)^{n-1}$.

(ii) $E^{(\alpha)}\big(\exp(-\beta d_{H_x^{(n)}})\big) = \big(\psi(\beta x)\big)^n \big(\theta(\beta x)\big)^n$.

(iii) $E^{(\alpha)}\big(\exp(-\beta g_{H_x^{(n)}})\big) = \exp(\beta x)\psi_n(\beta x)$.

4) Our aim in making the asymptotic study of $P^{(\alpha)}(V_{g_t}^1 \leqslant x_1, \ldots, V_{g_t}^n \leqslant x_n)$ was to penalise the Bessel process by the functional:

$$\Gamma_t^{(n)} = 1_{(V_{g_t}^1 \leqslant x_1, \ldots, V_{g_t}^n \leqslant x_n)}.$$

More precisely, we wish to show that, for any $\Lambda_s \in \mathcal{F}_s = \sigma\{R_u, u \leqslant s\}$, and for any $s \geqslant 0$:

$$Q(\Lambda_s) := \lim_{t\to\infty} \frac{E^{(\alpha)}(1_{\Lambda_s}\Gamma_t^{(n)})}{E^{(\alpha)}(\Gamma_t^{(n)})} = E^{(\alpha)}\big(1_{\Lambda_s} M_s^{(x_1,\ldots,x_n)}\big) \tag{42}$$

where $(M_s^{(x_1,\ldots,x_n)}, s \geqslant 0)$ is a $P^{(\alpha)}$-martingale, and then to describe the canonical process $(R_t, t \geqslant 0)$ under the probability Q induced by (42). We have been able to go through this study for $\alpha = 1/2$, and $n = 1$ in [8], that is, when $(R_t, t \geqslant 0)$ is reflecting Brownian motion, as well as in [9] for any $\alpha \in (0,1)$, and $n = 1$. We are presently investigating the general case ($\alpha \in (0,1)$, and $n \geqslant 1$).

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