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A sufficient condition for asymptotic normality of the normalized quadratic form $\Psi_n(f, g)$

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Abstract

Many sufficient conditions of asymptotic normality of the normalized quadratic form $\Psi_n(f, g)$ have been proposed since 1958. The less restrictive was given in the paper of L. Giraitis and D. Surgailis (1990). Using a linear operator approach, it is possible to produce an even less restrictive sufficient condition on the couple of functions (f, g). ***To cite this article: V. Solev, L. Gerville-Reache, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une condition suffisante de normalité asymptotique de la forme quadratique standardisée $\Psi_n(f, g)$. Des conditions suffisantes de normalité asymptotique de la forme quadratique standardisée $\Psi_n(f, g)$ se sont succédées depuis 1958. La moins restrictive fût proposée par L. Giraitis et D. Surgailis en 1990. En abordant le problème sous l'angle des opérateurs linéaires, il est possible de produire une condition suffisante encore moins restrictive sur le couple de fonctions (f, g). ***Pour citer cet article : V. Solev, L. Gerville-Reache, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit $X_t,\ t \in \mathbb{Z}$, un processus stationnaire Gaussien à valeurs réelles de moyenne zéro et de fonction de corrélation

$$r_{t-s} = \mathbf{E} X_t X_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, \mathrm{e}^{\mathrm{i}(t-s)u}\, \mathrm{d}u \tag{1}$$

où la fonction $f \in L^1$ est la densité spectrale de $X_t,\ t \in \mathbb{Z}$.

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Dans ce qui suit, on note $L^2(\mathrm{d}P)$ l'espace L^2-construit sur la mesure de probabilité P telle que $X_k \in L^2(\mathrm{d}P)$, $k \in \mathbb{Z}$. Pour un sous-espace fini $M \subset \mathbb{Z} \times \mathbb{Z}$ nous considerons la variable aléatoire J qui peut être représentée sous la forme suivante :

$$J = \sum_{(t,s)\in M} c_{t,s} X_t X_s - \mathbf{E}\left\{ \sum_{(t,s)\in M} c_{t,s} X_t X_s \right\},$$

et l'ensemble linéaire $\mathscr{L}_*$ de toutes les variables aléatoires J. Notons $\mathscr{L}$ la fermeture dans l'espace $L^2(\mathrm{d}P)$ de l'ensemble linéaire $\mathscr{L}_*$.

Maintenant, considérons la forme quadratique suivante

$$\Gamma_n(f,g) = \sum_{-n\leqslant t,s\leqslant n} g_{t-s} X_t X_s,$$

où la fonction $g \in L^1$ est définie par la relation $g(u) = \sum_{-\infty}^{\infty} g_k \,\mathrm{e}^{iku}$. Nous étudions le comportement asymptotique de la forme quadratique normalisée suivante :

$$\Psi_n(f,g) = \frac{\Gamma_n(f,g) - \mathbf{E}\Gamma_n(f,g)}{\sqrt{2n+1}}. \tag{2}$$

Ce problème a été étudié par de nombreux auteurs. La première fois, ce fût dans le livre de Grenander et Szegö [5]. Par la suite, Rosenblatt [8] et Ibragimov [6] prouvèrent la normalité asymptotique de $\Psi_n(f,g)$ sous les conditions : $f \in L^2$ and $g \in Ł^\infty$, et Avram [1] sous les conditions : $f \in L^p$, $g \in Ł^q$ and $1/p + 1/q \leqslant 1/2$. La moins restrictive des conditions suffisantes de normalité asymptotique a été suggerée par Giraitis and Surgailis [4]. Ils ont prouvé que si $f \in L^2$, $g \in Ł^2$, $fg \in L^2$, et

$$\int_{-\pi}^{\pi} f^2(u) g^2(u-v)\,\mathrm{d}u \to \int_{-\pi}^{\pi} f^2(u) g^2(u)\,\mathrm{d}u.$$

Alors $\Psi_n(f,g) \to N(0, \sigma^2(f,g))$ en distribution, quand $n \to \infty$, où

$$\sigma^2(f,g) = 4\pi \int_{-\pi}^{\pi} f^2(x) g^2(x)\,\mathrm{d}x. \tag{3}$$

Giraitis and Surgailis [4] ont émis l'hypothèse que cette asymptotique était vraie uniquement si $fg \in L^2$. Ginovian [2], Ginovian et Saakian [3] ont construit un contre exemple de cette hypothèse et ont donné une nouvelle condition locale de normalité asymptotique.

Il est simple de mettre en évidence que la valeur limite $\sigma^2(f,g)$ de la variation de $\Psi_n(f,g)$ est plus stable à la perturbation de la fonction g, qu'à celle de $\operatorname{Var}\Psi_n(f,g)$. Il est clair que pour f et n fixés, la forme quadratique $\Psi_n(f,g)$ peut être considerée comme un opérateur linéaire : $\Psi_n : g \mapsto \Psi_n(f,g)$. Aussi, dans le but de prouver la normalité asymptotique, il est suffisant d'étudier le problème suivant : pour quelles « boules » B_ε (éventuellement dépendantes de f)

$$\varlimsup_{n\to\infty} \sup_{g\in B_\varepsilon} \left\| \Psi_n(f,g) \right\|_{L^2(\mathrm{d}P)} \leqslant \varepsilon.$$

1. Introduction

Let X_t, $t \in \mathbb{Z}$, be a Gaussian real valued stationary process with zero mean and correlation function

$$r_{t-s} = \mathbf{E} X_t X_s = \mathbf{E} X_t \overline{X_s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\,\mathrm{e}^{\mathrm{i}(t-s)u}\,\mathrm{d}u, \tag{1}$$

and spectral density $f \in L^1$, $f(u) = f(-u)$.

Further we shall use the notation $L^2(\mathrm{d}P)$ for the L^2-space constructed on probability measure P such that $X_k \in L^2(\mathrm{d}P),\ k \in \mathbb{Z}$. For a finite subset $M \subset \mathbb{Z}\times\mathbb{Z}$ consider random variables J which may be represented in the following form:

$$J = \sum_{(t,s)\in M} c_{t,s} X_t X_s - \mathbf{E}\left\{\sum_{(t,s)\in M} c_{t,s} X_t X_s\right\},$$

and linear set $\mathscr{L}_*$ that consist of all such random variables J. Denote by $\mathscr{L}$ the closure in the space $L^2(\mathrm{d}P)$ of the linear set $\mathscr{L}_*$.

Now consider a quadratic form,

$$\Gamma_n(f,g) = \sum_{-n\leqslant t,s\leqslant n} g_{t-s} X_t X_s,$$

generated by an even function $g \in L^1,\ g(u) = \sum_{-\infty}^{\infty} g_k \,\mathrm{e}^{\mathrm{i}ku}$. We wish to investigate the asymptotic behavior of the normalized quadratic form,

$$\Psi_n(f,g) = \frac{\Gamma_n(f,g) - \mathbf{E}\Gamma_n(f,g)}{\sqrt{2n+1}}. \tag{2}$$

This problem was investigated by many authors. For the first time in the monograph of Grenander and Szegö [5]. Then Rosenblatt [8] and Ibragimov [6] proved asymptotical normality of $\Psi_n(f,g)$ under conditions $f \in L^2,\ g \in \text{Ł}^{\infty}$, and Avram [1] under conditions $f \in L^p,\ g \in \text{Ł}^q,\ 1/p + 1/q \leqslant 1/2$. The less restrictive sufficient condition for asymptotical normality was suggested by Giraitis and Surgailis [4]. They proved that if $f \in L^2,\ g \in \text{Ł}^2,\ fg \in L^2$, and

$$\int_{-\pi}^{\pi} f^2(u) g^2(u-v)\,\mathrm{d}u \to \int_{-\pi}^{\pi} f^2(u) g^2(u)\,\mathrm{d}u,$$

then $\Psi_n(f,g) \to N(0, \sigma^2(f,g))$ in distribution, as $n \to \infty$, where

$$\sigma^2(f,g) = 4\pi \int_{-\pi}^{\pi} f^2(x) g^2(x)\,\mathrm{d}x. \tag{3}$$

Giraitis and Surgailis [4] advanced the hypothesis that this statement is true only under condition $fg \in L^2$.

Ginovian [2], Ginovian and Saakian [3] constructed a counter-example for this hypothesis and gave new interesting local sufficient conditions for asymptotical normality.

It is easy to find out, that the limiting value $\sigma^2(f,g)$ for the variation of $\Psi_n(f,g)$ is more stable with respect to perturbation of function g, than $\operatorname{Var}\Psi_n(f,g)$. It is clear that for fixed f and n quadratic form $\Psi_n(f,g)$ may be considered as linear operator $\Psi_n : g \mapsto \Psi_n(f,g)$. So, in order to prove asymptotical normality it is sufficient to investigate the problem: for which 'balls' B_ε (may be depending on f)

$$\overline{\lim_{n\to\infty}} \sup_{g\in B_\varepsilon} \left\|\Psi_n(f,g)\right\|_{L^2(\mathrm{d}P)} \leqslant \varepsilon. \tag{4}$$

2. Spectral representation of quadratic forms

Denote by L^2_f the L^2-space on the interval $[-\pi,\pi]$ constructed on the measure with density $\frac{1}{2\pi} f$. Let $(\cdot,\cdot)_f$, $\|\cdot\|_f$ be respectively the inner product and the norm in the space L^2_f:

$$(\varphi,\psi)_f = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(u)\overline{\psi(u)} f(u)\,\mathrm{d}u, \qquad \|\varphi\|_f^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\varphi(u)\right|^2 f(u)\,\mathrm{d}u. \tag{5}$$

Consider the following spectral representation of process $X_t,\ t \in \mathbb{Z}$:

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}tu} Z(\mathrm{d}u), \tag{6}$$

where Gaussian orthogonal measure $Z(\mathrm{d}u)$ is defined by the relations

$$\mathbf{E}\left|\int_{-\pi}^{\pi}\varphi(u)Z(\mathrm{d}u)\right|^2=\|\varphi\|_f^2,\qquad \mathbf{E}\left\{\int_{-\pi}^{\pi}\varphi(u)Z(\mathrm{d}u)\right\}=0.$$

Denote by $L_f^2\times L_f^2$ for the Hilbert space (of functions $\psi:\mathbb{R}^2\to C$) with the inner product

$$(\psi_1,\psi_2)_{L_f^2\times L_f^2}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi_1(u,v)\overline{\psi_2(u,v)}f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v,$$

and the norm

$$\|\psi\|_{L_f^2\times L_f^2}^2=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\psi(u,v)\right|^2f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v.$$

It is clear that

$$J=J(\psi)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi(u,v)Z(\mathrm{d}u)\overline{Z(\mathrm{d}v)}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(u,u)f(u)\,\mathrm{d}u,\tag{7}$$

where the function ψ is defined by

$$\psi(u,v)=\sum_{(t,s)\in M}c_{t,s}\mathrm{e}^{\mathrm{i}(tu-sv)}.\tag{8}$$

The linear set of all such functions ψ will be denoted by $\mathscr{S}$.

Another form of representation (7) is known. Consider the stochastic measure $\Phi(\mathrm{d}u,\mathrm{d}v)$ defined by

$$\Phi(A,B)=Z(A)\overline{Z(B)}-\mathbf{E}\left\{Z(A)\overline{Z(B)}\right\}.$$

The stochastic integral

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi(u,v)\Phi(\mathrm{d}u,\mathrm{d}v)\tag{9}$$

is well defined on $\mathscr{S}$ and coincides with the 'stochastic integral' which is defined in (7). Further we shall use notation $J(\psi)$ for both 'stochastic integrals'.

It is easy to prove (see in details Ibragimov and Rozanov [7]) that the stochastic integral $J(\psi)$ which is defined as the linear operator $J:\mathscr{S}\to\mathscr{L}_*$, satisfies to the condition

$$\left\|J(\psi)\right\|_{L^2(\mathrm{d}P)}^2=2\|\psi\|_{L_f^2\times L_f^2}^2.\tag{10}$$

Since the linear set $\mathscr{S}$ is dense in $L_f^2\times L_f^2$, then the linear operator J may be continued to the operator $J:L_f^2\times L_f^2\to\mathscr{L}$, such that (10) is fulfilled.

It is easy to see that

$$\Gamma_n(f,g)=\sum_{-n\leqslant t,\ s\leqslant n}g_{t-s}X_tX_s=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi_n(u,v)Z(\mathrm{d}u)\overline{Z(\mathrm{d}v)},$$

where

$$\psi_n(u,v)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(y)\frac{\sin\frac{2n+1}{2}(u-y)}{\sin\frac{(u-y)}{2}}\times\frac{\sin\frac{2n+1}{2}(v-y)}{\sin\frac{(v-y)}{2}}\,\mathrm{d}y.\tag{11}$$

Thus, we obtain

$$\Gamma_n(f,g)-\mathbf{E}\Gamma_n(f,g)=J(\psi_n).\tag{12}$$

3. Main result

It is clear that for fixed f and n the quadratic form $\Psi_n(f,g)$ may be considered as linear operator $\Psi_n : g \mapsto \Psi_n(f,g)$. Simple and well-known arguments for the proof of asymptotic normality of $\Psi_n(f,g)$ consist in the following evident lemma.

Lemma 2. *Suppose $\mathscr{G}_0 \subset \mathscr{G} \subset L^1$ and*

(1) *for any $g_0 \in \mathscr{G}_0$ there exists the limit $\lim_{n\to\infty} \|\Psi_n(f,g_0)\|^2_{L^2(\mathrm{d}P)} = \sigma^2(f,g_0) < \infty$, and random variables $\Psi_n(f,g_0) \to N(0,\sigma^2(f,g_0))$ in distribution, as $n\to\infty$,*
(2) *for any $g \in \mathscr{G}$ and any $\varepsilon > 0$ there exists $g_0 \in \mathscr{G}_0$ (dependent on g and ε) such that*

$$\overline{\lim_{n\to\infty}} \left\|\Psi_n(f, g-g_0)\right\|_{L^2(\mathrm{d}P)} \leqslant \varepsilon.$$

Then if $g \in \mathscr{G}$ and $\sigma^2(f,g) < \infty$, then there exists the limit $\lim_{n\to\infty} \|\Psi_n(f,g)\|^2_{L^2(\mathrm{d}P)} = \sigma^2(f,g)$, and $\Psi_n(f,g) \to N(0,\sigma^2(f,g))$ in distribution, as $n \to \infty$.

Denote for a function $f \in L^1$ and an interval I with the length $|I|$

$$f_I = \frac{1}{|I|}\int\limits_I f(x)\,\mathrm{d}x.$$

Theorem. *Suppose that there exists a constant C such that for all intervals I*

$$f_I\big\{|g|\big\}_I \leqslant C\big(\big\{f|g|\big\}_I + 1\big)$$

and

$$\sigma^2(f,g) = 4\pi \int\limits_{-\pi}^{\pi} f^2(x)g^2(x)\,\mathrm{d}x < \infty.$$

Then $\Psi_n(f,g) \to N(0,\sigma^2(f,g))$ in distribution, as $n\to\infty$.

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References

[1] F. Avram, On Bilinear Forms in Gaussian Random Variables and Toeplitz Matrices, University of California Press, Berkeley and Los Angeles, 1958.
[2] M.S. Ginovyan, On Toeplitz type quadratic forms in Gaussian stationary process, Probab. Theory Related Fields 100 (1994) 395–406.
[3] M.S. Ginovyan, A.A. Saakian, On central limit theorem for Toeplitz quadratic forms in Gaussian stationary sequences, Theory Probab. Appl. (2005) submitted for publication.
[4] L. Giraitis, D. Surgailis, A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotical normality of Whittle's estimate, Probab. Theory Related Fields 86 (1990) 87–104.
[5] U. Grenander, G. Szegö, Toeplitz Form and Their Applications, University of California Press, Berkeley and Los Angeles, 1958.
[6] I.A. Ibragimov, On estimation of spectral density of stationary Gaussian process, transform, Theory Probab. Appl. 8 (1963) 391–430.
[7] I.A. Ibragimov, Yu.A. Rozanov, Gaussian Processes, Mir, Moscow, 1974.
[8] M. Rosenblatt, Asymptotic behavior of eigenvalues of Toeplitz form, transform, J. Math. Mech. 11 (1962) 941–950.