Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 342 (2006) 859–864
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

A direct method for the stabilization of some locally damped semilinear wave equations

Louis Tcheugoué Tébou

Department of Mathematics, Florida International University, Miami, FL 33199, USA

Received and accepted 30 March 2006

Available online 5 May 2006

Presented by Haïm Brezis

Abstract

First, we consider a semilinear wave equation with a locally distributed damping in a bounded domain. Using the Carleman estimate, we devise an elementary proof of the exponential decay of the energy of this system. Afterwards we apply the same technique to the stabilization of the same type of equation in the whole space. Our proofs are constructive, and much simpler than those found in the literature. ***To cite this article: L. Tcheugoué Tébou, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une méthode directe pour la stabilisation de quelques équations des ondes semi-linéaires localement amorties. Dans un premier temps, nous considérons une équation des ondes semi-linéaire avec un amortissement localement distribué dans un domaine borné. A l'aide de l'inégalité de Carleman, nous construisons une preuve élémentaire et directe de la décroissance exponentielle de l'énergie de ce système. Par la suite, nous appliquons la même technique pour étudier la stabilisation du même type d'équation dans l'espace tout entier. Nos démonstrations sont constructives, et beaucoup plus simples que celles existantes. ***Pour citer cet article : L. Tcheugoué Tébou, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit Ω un ouvert borné de $\mathbb{R}^d$ de classe C^2. Soit $f:\mathbb{R}\to\mathbb{R}$ une fonction dérivable satisfaisant les conditions de croissance habituelles (voir (1)), et soit $a\in L^\infty(\Omega)$ une fonction positive vérifiant (2). On considère le système des ondes amorti (voir (3)).

L'un de nos objectifs dans cette note est de démontrer le résultat suivant :

Théorème 1. *Soient $y^0\in H_0^1(\Omega)$ et $y^1\in L^2(\Omega)$. Soit ω un voisinage de $\partial\Omega$. On suppose que f vérifie* (1) *et que a satisfait* (2). *Soit E_0 une constante strictement positive quelconque. On suppose de plus l'alternative suivante* : *ou bien f est globalement lipschitzienne, ou bien $E(0)\leqslant E_0$, où $E(0)$ est l'énergie initiale. Alors il existe des constantes*

E-mail address: teboul@fiu.edu (L. Tcheugoué Tébou).

1631-073X/$ – see front matter © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2006.04.010

strictement positives M et α, dépendant éventuellement de E_0 telles que l'énergie E de chaque solution de (3) *vérifie* :

$$E(t) \leqslant M\big[\exp(-\alpha t)\big]E(0), \quad \forall t \geqslant 0. \tag{$*$}$$

Remarque 1. Notons que les constantes M et α sont indépendantes des données initiales lorsque f est globalement lipschitzienne ; de sorte que la stabilité exponentielle est uniforme dans l'espace d'énergie. Lorsque f n'est pas globalement lipschitzienne, la stabilité exponentielle est uniforme seulement sur toute boule dans l'espace d'énergie. Notre méthode ne nous a pas permis de retrouver la stabilité exponentielle uniforme de [17] pour les nonlinéarités sur-linéaires satisfaisant pour un certain δ strictement positif, $sf(s) \geqslant (2+\delta)F(s)$ pour tout s. Mais elle a l'avantage d' être constructive, et de permettre de simplifier de façon significative les démonstrations existantes basées sur des arguments de compacité-unicité.

Remarque 2. Notre méthode de démonstration nous permet aussi d'établir des résultats similaires pour l'équation des ondes dans des domaines non bornés. Nous proposons ci-dessous un exemple modèle.

La fonction f étant donnée comme ci-dessous, on considère l'équation semi-linéaire des ondes amortie (voir (7)).

Pour ce nouveau système, nous démontrons le résultat de décroissance exponentielle :

Théorème 2. *Soient $y^0 \in H_0^1(\Omega)$ et $y^1 \in L^2(\Omega)$. Supposons que $\omega = \{x \in \mathbb{R}^d;\ |x| > L\}$ pour un certain $L > 0$. On suppose que f vérifie* (1) *et que a satisfait* (2). *Soit E_0 un nombre réel strictement positif. On suppose en outre l'alternative suivante* : *ou bien f est globalement lipschitzienne, ou bien $\widetilde{E}(0) \leqslant E_0$, où $\widetilde{E}(0)$ est l'énergie initiale. Alors l'énergie $\widetilde{E}$ de chaque solution de* (7) *vérifie une estimation semblable à* $(*)$.

1. Position of the problem and statements of main results

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, with a boundary of class C^2. Let $f:\mathbb{R} \to \mathbb{R}$ be a differentiable function which satisfies the following growth conditions

$$\begin{aligned} &sf(s) \geqslant 0, \quad \forall s \in \mathbb{R},\\ &\exists C_1 > 0\colon \big|f'(s)\big| \leqslant C_1\big(1+|s|^{q-1}\big), \quad \forall s \in \mathbb{R}, \end{aligned} \tag{1}$$

where $q \geqslant 1$, $(d-2)q \leqslant d$. Let $a \in L^\infty(\Omega)$ be a nonnegative function satisfying

$$\exists a_0 > 0\colon a(x) \geqslant a_0, \quad \text{a.e. } x \in \omega, \tag{2}$$

where ω is a neighborhood of $\partial\Omega$, that is to say, the intersection of Ω and a neighborhood of $\partial\Omega$. Consider the damped wave equation

$$\begin{cases} y'' - \Delta y + p(x)y + f(y) + ay' = 0 & \text{in } \Omega \times (0,\infty),\\ y = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,\infty),\\ y(0) = y^0; \quad y'(0) = y^1 & \text{in } \Omega, \end{cases} \tag{3}$$

where $p \in L_+^m(\Omega)$, ($m = 2$ for $d = 1$, $m > 2$ for $d = 2$, and $m \geqslant d$ for $d \geqslant 3$), and $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. It is well known (cf. e.g. [9]) that under the above hypotheses on the data, system (3) has a unique weak solution $y \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$.

Consider the energy

$$E(t) = \frac{1}{2}\int_\Omega \big\{\big|y'(x,t)\big|^2 + \big|\nabla y(x,t)\big|^2 + p(x)\big|y(x,t)\big|^2\big\}\,dx + \int_\Omega F\big(y(x,t)\big)\,dx, \tag{4}$$

where $F(s) = \int_0^s f(r)\,dr$. It is easy to check that the energy E is a nonincreasing function of the time variable t. In particular we have the dissipation law:

$$E'(t) = -\int_\Omega a|y'|^2\,dx. \tag{5}$$

One of our objectives in this Note is to prove that E decays exponentially to zero. Our first result states as follows:

Theorem 1.1. *Let $y^0 \in H_0^1(\Omega)$ and $y^1 \in L^2(\Omega)$. Let ω be a neighborhood of $\partial\Omega$. Assume that the function f satisfies the growth conditions* (1)*, and the function a satisfies* (2)*. Let E_0 be a positive constant. Further assume the following alternative: either f is globally Lipschitz, or else $E(0) \leqslant E_0$. Then there exist positive constants M and α, possibly depending on E_0, such that the energy E of each solution of* (3) *satisfies*:

$$E(t) \leqslant M\big[\exp(-\alpha t)\big]E(0), \quad \forall t \geqslant 0. \tag{6}$$

Remark 1.2. It is worth noting that the constants M and α do not depend on the initial data when f is globally Lipschitz; so that in this case the exponential decay of the energy is uniform in the energy space. However, when f is not globally Lipschitz, the decay is uniform only on every ball in the energy space.

Remark 1.3. This Note was motivated by earlier works of Zuazua on this topic [17,18]; in [17], the author provides two different proofs:

– one for globally Lipschitz nonlinearities f satisfying either $\lim_{s\to-\infty} f'(s)$ and $\lim_{s\to\infty} f'(s)$ both exist, or $\lim_{|s|\to\infty} \frac{f(s)}{s}$ exists; it is easy to check that this condition excludes functions such as $f(s) = \beta s \sin^2(\ln(1+s^2))$, $(\beta > 0)$, that are globally Lipschitz but for which none of the aforementioned limits exist,
– one for superlinear functions f that further satisfy $sf(s) \geqslant (2+\delta)F(s)$ for some $\delta > 0$.

All the proofs in [17,18] are based on the unique continuation property of Ruiz [12], and they lead to uniform exponential decay estimates of the energy. Later on Dehman [2] reduced the two proofs in [17] to a single proof, but for bounded initial data. Subsequently the results of [17,18,2] were improved to include all the subcritical nonlinearities f by Dehman, Lebeau and Zuazua in [3]. It is also of interest to mention Nakao's paper [11], where the author discusses the same type of questions for systems involving nonlinearities of the form $f(x,s)$ – that are bounded in x – and nonlinear damping locally distributed on a neighborhood of a suitable subset of the boundary; he establishes polynomial and exponential energy decay estimates for small enough initial data. All the proofs in [2,3,11,17,18] are based on a compactness-uniqueness argument, and consequently do not lead to explicit decay rates. Our constructive approach, to be developed below, enables us to provide, in a single proof, uniform exponential decay estimates of the energy for globally Lipschitz nonlinearities f and local stabilization for all other functions f satisfying (1).

Concerning linear problems ($f(s) = ms, m \geqslant 0$) with linear or nonlinear dissipations, we refer the reader to (e.g. [1,5,7,10,13–15]). Our method is flexible, and it applies also to problems in unbounded domains. We now discuss a model example. Consider the damped semilinear wave equation:

$$\begin{cases} y'' - \Delta y + y + f(y) + ay' = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ y(0) = y^0, \quad y'(0) = y^1 & \text{in } \mathbb{R}^d \end{cases} \tag{7}$$

where $y^0 \in H^1(\mathbb{R}^d)$, $y^1 \in L^2(\mathbb{R}^d)$, the function f is given as above while now the nonnegative function a lies in $L^\infty(\mathbb{R}^d)$, and satisfies

$$\exists a_0 > 0\colon a(x) \geqslant a_0, \quad \text{a.e. } x \in \omega = \big\{x \in \mathbb{R}^d;\ |x| > L\big\} \tag{8}$$

for some $L > 0$.

For this new system, the energy given by

$$\widetilde{E}(t) = \frac{1}{2}\int_{\mathbb{R}^d} \big\{\big|y'(x,t)\big|^2 + \big|\nabla y(x,t)\big|^2 + \big|y(x,t)\big|^2\big\}\,\mathrm{d}x + \int_{\mathbb{R}^d} F\big(y(x,t)\big)\,\mathrm{d}x, \tag{9}$$

is also a nonincreasing function of the time variable. More precisely, we have:

Theorem 1.4. *Let $y^0 \in H^1(\mathbb{R}^d)$ and $y^1 \in L^2(\mathbb{R}^d)$. Set $\omega = \{x \in \mathbb{R}^d;\ |x| > L\}$ for some $L > 0$. Assume that f satisfies* (1) *and that for a, condition* (8) *holds. Let E_0 be a positive constant. Further assume the following alternative: either f is globally Lipschitz, or else $\widetilde{E}(0) \leqslant E_0$. Then there exist positive constants M and α, possibly depending on E_0, such that the energy E of each solution of* (7) *satisfies*:

$$\widetilde{E}(t) \leqslant M\big[\exp(-\alpha t)\big]\widetilde{E}(0), \quad \forall t \geqslant 0. \tag{10}$$

Remark 1.5. As was the case for Theorem 1.1, when f is globally Lipschitz, the exponential decay is uniform in the energy space while for other functions the exponential decay is uniform only on every ball in the energy space.

2. Basic ideas for proving Theorems 1.1 and 1.4

For the sequel we need the following notations: let $\mu > 0$, $\eta > 1$, and for $(x,t) \in \mathbb{R}^{d+1}$, set $\varphi(x,t) = \eta^2 t^2 - |x|^2$, and $D^\mu = \{(x,t) \in \mathbb{R}^{d+1};\ \varphi(x,t) > \mu\}$. The proofs of these theorems are based on the following Carleman estimate due to Ruiz [12, Proposition 1]:

Lemma 2.1. *Let K be a compact subset of D^μ, then there exists a $\lambda_0 > 0$ and a constant $C = C(K,\mu)$, independent of u and λ such that for any $\lambda > \lambda_0$ and $u \in C_0^\infty(K)$ we have*

$$\lambda \left\| e^{2\lambda\varphi} u \right\|_{L^2(K)}^2 \leqslant C \left\| e^{2\lambda\varphi} \Box u \right\|_{H^{-1}(K)}^2, \tag{11}$$

where $\Box u = u_{tt} - \Delta u$.

Furthermore estimate (11) *holds for all $u \in L^2(K)$ such that $\Box u \in H^{-1}(K)$.*

2.1. Sketch of the proof of Theorem 1.1

It suffices to show that there exist $T > 0$, and $C > 0$ such that

$$E(T) \leqslant C \int_0^T \int_\Omega a|y'|^2 \, dx \, dt. \tag{12}$$

We emphasize that the constant C does not depend on the initial data for globally Lipschitz functions f, while C depends on E_0^{q-1} for other functions. In either case, from (12), we derive

$$E(T) \leqslant \frac{C}{C+1} E(0),$$

so that using the semigroup property, we get (6) with $M = (C+1)/C$, and $\alpha = \log((C+1)/C)/T$. So it remains to prove (12). From now on, C denotes various positive constants independent of the initial data, and λ unless specified otherwise. Set $T_0 = \operatorname{diam}(\Omega \setminus \omega) + \inf\{|x|;\ x \in \Omega \setminus \omega\}$. Let $T_1 > T_0$. Then there exists $\mu > 0$ such that $T_1^2 > T_0^2 + \mu$. Let $\eta > 1$. For every $t \geqslant T_1$, and every $x \in \overline{\Omega \setminus \omega}$, we have $\eta^2 t^2 - |x|^2 > \mu$, so that for each $T > T_1$, the set $K = \overline{\Omega \setminus \omega} \times [T_1, T]$ is a compact subset of D^μ. Further, if we set $u = y'$, then $u \in L^2(K)$, and $\Box u \in H^{-1}(K)$. Now $\Box u = -pu - f'(y)u - au_t$. Applying Lemma 2.1, we obtain

$$\lambda \left\| e^{2\lambda\varphi} u \right\|_{L^2(K)}^2 \leqslant C \left\| e^{2\lambda\varphi} \bigl(-pu - f'(y)u - au_t\bigr) \right\|_{H^{-1}(K)}^2. \tag{13}$$

Now by Hahn–Banach theorem, Sobolev embedding theorem, and Hölder inequality we find

$$\begin{aligned} &\left\| e^{2\lambda\varphi} pu \right\|_{H^{-1}(K)} \leqslant C \left\| e^{2\lambda\varphi} u \right\|_{L^2(K)}, \\ &\left\| e^{2\lambda\varphi} f'(y)u \right\|_{H^{-1}(K)} \leqslant C \left\| e^{2\lambda\varphi} u \right\|_{L^2(K)} \bigl(1 + \|\nabla y\|_{L^2(\Omega\times(T_1,T))}^{q-1}\bigr). \end{aligned} \tag{14}$$

On the other hand, applying Hahn–Banach theorem, and integration by parts over K, we get

$$\left\| e^{2\lambda\varphi} au_t \right\|_{H^{-1}(K)} \leqslant C(1+\lambda) \left\| e^{2\lambda\varphi} au \right\|_{L^2(K)}. \tag{15}$$

When f is globally Lipschitz and λ is large enough, a combination of (13)–(15), and (2) yields

$$\|y'\|_{L^2(\Omega\times(T_1,T))}^2 \leqslant C(\lambda) \int_{T_1}^T \int_\Omega a|y'|^2 \, dx \, dt. \tag{16}$$

For other functions f, we have instead

$$\|y'\|_{L^2(\Omega\times(T_1,T))}^2 \leqslant C\bigl(\lambda, E_0^{q-1}\bigr) \int_{T_1}^T \int_\Omega a|y'|^2 \, dx \, dt. \tag{17}$$

At this stage, we note that we will be finished once we have proven that

$$E(T) \leqslant C_0 \int_{T_1}^{T} \int_{\Omega} |y'|^2 \, dx \, dt, \tag{18}$$

for some positive constant C_0 which may or may not depend on the initial data.

To this end, let $r \in \mathcal{C}^1([T_1, T])$ with $r(T) = r(T_1) = 0$. Multiplying the first equation in (3) by ry and integrating by parts over $\Omega \times [T_1, T]$, we get

$$2\int_{T_1}^{T} r(t)\mathcal{E}(t)\, dt + \int_{T_1}^{T}\int_{\Omega} ryf(y)\, dx\, dt = 2\int_{T_1}^{T}\int_{\Omega} r|y'|^2\, dx\, dt + \int_{T_1}^{T}\int_{\Omega} r'y'y\, dx\, dt - \int_{T_1}^{T}\int_{\Omega} ray'y\, dx\, dt, \tag{19}$$

where $\mathcal{E}(t) = \frac{1}{2}\int_{\Omega}\{|y'(x,t)|^2 + |\nabla y(x,t)|^2 + p(x)|y(x,t)|^2\}\, dx$. We observe that for globally Lipschitz f, the energies E and $\mathcal{E}$ are trivially equivalent while for other functions, we have

$$\mathcal{E}(t) \leqslant E(t) \leqslant C\big(1 + E_0^{(q-1)/2}\big)\mathcal{E}(t). \tag{20}$$

One easily derives (18) from (19), (20), (1), and the fact that the energy E is nonincreasing.

2.2. *Sketch of the proof of Theorem 1.4*

Following the same steps as in the sketch of the proof of Theorem 1.1, one is led to

$$\widetilde{E}(T) \leqslant C_1 \int_{T_1}^{T}\int_{\mathbb{R}^d} a\big|y'(x,t)\big|^2\, dx\, dt, \tag{21}$$

from which one easily derives the claimed estimate.

Remark 2.2. As one can see, the merit of our method is that it provides a much simpler proof than those found in the literature [2,3,11,17,18]. One may allow the product $sf(s)$ to be negative; namely '$f(s)s \geqslant -bs^2$, $\forall s \in \mathbb{R}$' for some small enough positive constant b. Our constructive approach is much more interesting for globally Lipschitz nonlinearities f because in this case, it enables us to improve in some way all the earlier results.

Remark 2.3. Our approach also enables us to deal with the more general problem

$$\begin{cases} y'' - \Delta y + f(x,y) + ag(y') = 0 & \text{in } \Omega \times (0,\infty), \\ y = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,\infty), \\ y(0) = y^0; \quad y'(0) = y^1 & \text{in } \Omega, \end{cases} \tag{22}$$

where Ω may be bounded or unbounded, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and satisfies '$g(s)s \geqslant cs^2$, $\forall s \in \mathbb{R}$' for some positive constant c, and $f(x,s)$ is measurable in x and differentiable in s with

$$\begin{aligned} &sf(x,s) \geqslant 0, \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \\ &\big|f_{,s}(x,s)\big| \leqslant C\big(p(x) + |s|^{q-1}\big), \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \end{aligned} \tag{23}$$

where for bounded domains, p is given as above, and for unbounded domains, $p \in L^m_+(\Omega)$, ($p \in L^1_+(\Omega) \cap L^2_+(\Omega)$ for $d = 1$, $m > 2$ for $d = 2$, and $m \geqslant d$ for $d \geqslant 3$), and p further satisfies an inequality of type (8). The latter condition on p for the case of unbounded domains is used to ensure the coercivity of the associated energy. Observe that when $\Omega = \mathbb{R}^d$, no boundary conditions are needed. Also, for proper unbounded open subsets Ω, the function a shall generally satisfy both (2) and (8) in order for the exponential decay to hold [1]; otherwise the geometric control condition of Bardos, Lebeau and Rauch may fail thereby precluding the exponential decay of the energy.

Remark 2.4. Our method, though flexible, has some drawbacks among which the stringent geometry restriction that the damping be located in a neighborhood of the whole boundary (for bounded or exterior domains); this is a consequence of the Ruiz Carleman inequality on which our constructive approach is built. We note that other explicit Carleman inequalities exist in the literature [4], that might enable us to improve on the geometry restriction by allowing for damping located in a neighborhood of a suitable subset of the boundary; this and more will be examined in [16]. We also note that our method does not seem to allow for the use of more general nonlinear dampings as in (e.g. [6,8,10,11,14]). In the case of the whole space, it would be interesting to find out whether some of our ideas could be combined with Strichartz estimates to recover some of the results in [3]. The extension of our approach to, say, the plate equation with clamped boundary conditions is wide open.

Acknowledgements

The author thanks the referee for the kind comments and suggestions provided.

References

[1] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.
[2] B. Dehman, Stabilisation pour l'équation des ondes semi-linéaire, Asymptotic Anal. 27 (2001) 171–181.
[3] B. Dehman, G. Lebeau, E. Zuazua, Stabilization and control for the subcritical semilinear wave equation, Ann. Sci. École Norm. Sup. (4) 36 (4) (2003) 525–551.
[4] T. Duyckaerts, X. Zhang, E. Zuazua, On the optimality of the observability for parabolic and hyperbolic systems with potentials, in press.
[5] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps, Portugal. Math. 46 (1989) 245–258.
[6] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, RAM, Masson & John Wiley, Paris, 1994.
[7] J. Lagnese, Control of wave processes with distributed control supported on a subregion, SIAM J. Control Optim. 21 (1983) 68–85.
[8] I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, Differential Integral Equations 6 (1993) 507–533.
[9] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod–Gauthier–Villars, Paris, 1969.
[10] P. Martinez, PhD Thesis, University of Strasbourg, 1998.
[11] M. Nakao, Global and periodic solutions for nonlinear wave equations with some localized nonlinear dissipation, J. Differential Equations 190 (1) (2003) 81–107.
[12] A. Ruiz, Unique continuation for weak solutions of the wave equation plus a potential, J. Math. Pures Appl. 71 (1992) 455–467.
[13] L.R. Tcheugoué Tébou, Estimations d'énergie pour l'équation des ondes avec un amortissement nonlinéaire localisé, C. R. Acad. Paris, Ser. I 325 (1997) 1175–1179.
[14] L.R. Tcheugoué Tébou, Stabilization of the wave equation with localized nonlinear damping, J. Differential Equations 145 (1998) 502–524.
[15] L.R. Tcheugoué Tébou, Well-posedness and energy decay estimates for the damped wave equation with L^r localizing coefficient, Comm. Partial Differential Equations 23 (1998) 1839–1855.
[16] L.R. Tcheugoué Tébou, A Carleman estimate based approach for the stabilization of some locally damped semilinear wave equations, in preparation.
[17] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Comm. Partial Differential Equations 15 (1990) 205–235.
[18] E. Zuazua, Exponential decay for the semilinear wave equation with localized damping in unbounded domains, J. Math. Pures Appl. 15 (1990) 205–235.