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Partial Differential Equations

# Strong solutions of the Boltzmann equation in one spatial dimension

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**Abstract**

For the Boltzmann equation, the setting of a narrow shock tube implies that solutions $f(x, v, t)$ depend upon $v \in \mathbb{R}^3$, however they have one-dimensional spatial dependence. This Note discusses the case in which solutions are periodic in $x$, with controlled total energy and entropy, and such that the macroscopic density determined by the initial data is bounded. Our principal result is that the macroscopic density then remains bounded at all subsequent times, that is, this data gives rise to strong solutions which exist globally in time. Through a weak/strong uniqueness principle, these solutions are unique among the class of dissipative solutions. Additionally, we show that the flow of the Boltzmann equation propagates the moments in $v \in \mathbb{R}^3$ and derivatives in both $x_1 \in \mathbb{R}^1$ and $v \in \mathbb{R}^3$ of the solution $f(x, v, t)$. Our main theorems are valid for Boltzmann collision kernels which are bounded, and which have a relative velocity cutoff. The proofs depend upon a new averaging property of the collision operator and integral inequalities based in turn on entropy and on the Bony functional. ***To cite this article: A. Biryuk et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Les solutions globales de l'équation de Boltzmann dans la géometrie uni-dimensionnelle.** Dans un domaine qui représente un tube à choc, les solutions $f(x, v, t)$ de l'équation de Boltzmann dépendent de $v \in \mathbb{R}^3$ mais elles ne dépendent que de $x_1 \in \mathbb{R}^1$. Dans cette Note, on considère le cas de solutions périodiques en $x_1 \in \mathbb{R}^1$, dont la densité macroscopique initiale est finie, et l'énergie et l'entropie totales sont bornées par une certaine constante C. Le résultat principal est que la densité macroscopique de la solution reste bornée pour tout temps $t > 0$, c'est-à-dire, les conditions initiales donnent lieu à des solutions fortes qui existent globalement en temps. Le résultat implique l'unicité de nos solutions dans la classe de solutions dissipatives faibles. Ces solutions $f(x, v, t)$ conservent les propriétés de régularité en $x$ et en $v$, et les moments finis en $v$. Les théorèmes principaux sont valables pour des noyaux de collision de Boltzmann bornés, et avec une troncature de vitesse relative. Les démonstrations dépendent d'une propriété nouvelle de moyennisation de l'opérateur de collision, et de deux inégalités intégrales basées sur l'entropie et sur la fonctionnelle de Bony. ***Pour citer cet article : A. Biryuk et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


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On considère des solutions de l'équation de Boltzmann telles que la densité macroscopique est bornée en $x \in \mathbb{R}^1$. Il est connu que ces solutions possèdent de bonnes propriétés pour le problème de Cauchy, et elles satisfont un principe d'unicité faible/fort. Par contre, il y a peu de résultats en général sur l'existence globale de ces solutions. Dans cette Note nous montrons que dans le cas où $f_0 = f_0(x_1, v) : \mathbb{T}^1_x \times \mathbb{R}^3_v \to \mathbb{R}^+$ il existe des solutions dont la densité macroscopique est $L^\infty_x$ pour tout temps, étant données des conditions initiales $f_0(x_1, v)$ dont l'énergie et l'entropie totales sont finies, et étant données certains bornes sur la norme $L^\infty_x$ de $f_0$. L'équation de Boltzmann s'écrit (1), et l'opérateur de collision classique est donné par l'expression (2). La densité macroscopique est définie dans (3), les integrales $A(f)$ et $E(f)$ de la solution en $(x, v)$ sont conservées par le flot de l'équation (1), et $H(f|M)$, l'entropie relative par rapport au Maxwellian $M(v)$, est une fonction décroissante du temps.

L'hypothèse qu'on impose sur le noyau de collision de Boltzmann $K$ est qu'il est borné, et que certaines collisions entre particules n'influencent pas leurs trajets. Plus précisement, pour le premier théorème, on impose des bornes sur les collisions entre deux particules à grande vitesse relative (H1), tandis qu'on impose une deuxième hypothèse (H2) sur le noyau $K$ pour le deuxième résultat.

Une *solution forte* de l'équation de Boltzmann sur un intervalle $I = [0, T)$ est une distribution $f(x, v, t) : I \to L^\infty_x(L^1_v)$ qui satisfait l'équation (1), ce qui implique que la densité macroscopique $\rho(x, t)$ est dans $L^\infty_x$ pour tout temps $t \in I$. Pour la démonstration de nos deux théorèmes principaux, il est utile d'imposer une norme plus restreinte, ce qu'il est précisé dans la Définition 2.2. L'espace de Banach défini par cette norme est noté $X$, et on constate que l'espace $X$ est invariant par l'évolution de l'équation de Boltzmann linéairisée autour du vide, tandis que l'espace $L^\infty_x$ ne l'est pas.

Notre approche à la question des solutions fortes de l'équation de Boltzmann est basée sur l'expression (7) provenant du principe de Duhamel, et en éliminant le terme $Q^-$ dans l'opérateur de collision. Dans le terme $Q^+$, l'integrale sur la sphère implique un effet de moyennisation dans la variable spatiale $x_1 \in \mathbb{T}^1$, ce qui permet de déduire, en utilisant la décroissance de l'entropie en temps, une inégalité intégrale (8) pour la norme $\|f(\cdot, t)\|_X \leqslant \varphi(t)$. La question de l'existence des solutions globales fortes de Boltzmann se réduit à une question d'existence de la solution $\varphi(t)$ de l'inégalité (8) pour tout temps, ce qui dépende uniquement de la constante $\alpha := C_0 K_0 H(f|M)$.

**Théorème 0.1.** *On considère l'équation de Boltzmann* (1) *dont le noyau de collision* (2) *satisfait l'hypothèse* (H1). *Étant donné une condition initiale* $f_0(x, v) \geqslant 0$ *telle que* $f_0 \in X$ *et* $\alpha \leqslant 1$, *alors il existe une solution forte* $f(x, v, t)$ *pour tout* $t \in \mathbb{R}^+$.

Quand $\alpha > 1$, les solutions fortes existent au moins pour un intervalle de temps $[0, T)$, quantifié par $T = N/\|f_0\|_X^{1/n}$. Avec l'hypothèse (H2) sur le noyau de collision $K$, on déduit une deuxième inégalité intégrale (9) pour la norme $\|f(\cdot, t)\|_X$, en utilisant la fonctionnelle de Bony [3,4,13]. Le deuxième résultat est global en temps, à partir de données initiales $f_0 \in X$ telles que l'énergie et la masse totales soient finies.

**Théorème 0.2.** *On considère les solutions de l'équation de Boltzmann où le noyau de collision satisfait* (H1) *et* (H2). *Si les données initiales* $f_0(x, v) \geqslant 0$ *satisfont* $\|f_0\|_X < +\infty$ *et* $A(f_0) + E(f_0) < +\infty$ *alors il existe une solution forte globalement en temps.*

L'existence d'une solution forte implique, par le principe d'unicité faible/fort, que des solutions faibles construites par éventuellement d'autre moyens coïncident avec la solution forte. Ce qui est le cas pour les solutions dissipatives dans le sens de P.-L. Lions [11]. Les propriétés de base du flot de l'équation de Boltzmann impliquent que les moments en $v$ et les dérivées en $x$ et en $v$ de la solution se propagent en temps, pour les données initiales satisfaisant la condition du Théorème 2.5 sur les moments et les dérivées initiaux.

## 1. Introduction

In this Note we are concerned with solutions of the Boltzmann equation that propagate certain regularity properties, such as $L^\infty$ estimates of the macroscopic mass density, globally in time. We refer to these as *strong* solutions; a precise definition is given below. Solutions in this class have good properties of evolution with respect to the flow determined by the Boltzmann equation; they satisfy a weak/strong uniqueness principle, and they propagate estimates of moments and higher derivatives of the solution if these are available from the initial data. The most serious issue is however the global existence in time of such solutions. The principal results to date are valid either for short time intervals or for perturbations of certain special solutions such as the vacuum state, a global thermodynamical equilibrium, or spatially homogeneous states; reviews of this literature appear in [8,14]. The main results of the present paper concern the global existence of strong solutions $f(x,v,t)$, for which $f$ is constant in $(x_2,x_3)$, depending only on $(x_1,v,t)\in\mathbb{R}^1_x\times\mathbb{R}^3_v\times\mathbb{R}^+_t$.

The initial value problem for the Boltzmann equation can be formulated as

$$\partial_t f + v\cdot\nabla_x f = Q(f,f),\quad f(x,v,0)=f_0(x,v). \tag{1}$$

The solution $f(x,v,t)$ describes the phase space density of particles of a gas, hence we ask that $f\geqslant 0$. The nonlinear term $Q$ is the collision operator of Maxwell [12] and Boltzmann [2], given by the expression

$$\begin{aligned} Q(f,f)(x,v,t) &= \int_{\mathbb{R}^3_{v_*}}\int_{\mathbb{S}^2_\sigma}\big(f(v')f(v'_*)-f(v)f(v_*)\big)K\left(|v-v_*|,\frac{(v-v_*)\cdot\sigma}{|v-v_*|}\right)\mathrm{d}S_\sigma\,\mathrm{d}v_* \\ &= Q^+(f,f)(x,v,t)-Q^-(f,f)(x,v,t), \end{aligned} \tag{2}$$

where for brevity we sometimes suppress the arguments $x$ and $t$ in $f$. The velocities appearing in the arguments of $f$ satisfy the scattering relations $v+v_*=v'+v'_*$ and $(v'-v'_*)/|v-v_*|=\sigma\in\mathbb{S}^2$. The scattering density is determined by the Boltzmann collision kernel $K$, of which more will be said below.

The macroscopic *density* associated with a solution $f$ is given by

$$\rho(x,t)=\int_{\mathbb{R}^3_v} f(x,v,t)\,\mathrm{d}v. \tag{3}$$

We consider solutions which satisfy periodic boundary conditions $f(x+\gamma,v,t)=f(x,v,t)$, $\gamma\in\mathbb{Z}^3$, which is to say that $x\in\mathbb{T}_x:=\mathbb{R}^3_x/\mathbb{Z}^3$. We further restrict $f(x,v,t)$ to be independent of $(x_2,x_3)$. It is a different character of problem than the case where $x\in\mathbb{R}^3$ with $f\in L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)$, as in the present setting solutions do not have the possibility to disperse into the surrounding region of vacuum. The *total mass* and *total energy* of $f$ are defined respectively by the integrals

$$A(f)=\int_{\mathbb{T}_x}\rho(x,t)\,\mathrm{d}x,\qquad E(f)=\int_{\mathbb{T}_x}\int_{\mathbb{R}^3_v} f(x,v,t)|v|^2\,\mathrm{d}v\,\mathrm{d}x. \tag{4}$$

The *relative entropy* of $f$ with respect to the Maxwellian $M(v)$ is given by

$$H(f|M)=\int_{\mathbb{T}_x}\int_{\mathbb{R}^3_v} f\log\left(\frac{f}{M}\right)-f+M\,\mathrm{d}v\,\mathrm{d}x,\quad \text{where } M(v)=\frac{a}{\sqrt{2\pi b}^3}\,\mathrm{e}^{-|v-c|^2/2b}, \tag{5}$$

for $a,b>0$ and $c\in\mathbb{R}^3$. The Maxwellian functions $f(x,v,t)=M(v)$ are the equilibrium solutions of (1). We remark that if $H(f|M)$ is finite for some Maxwellian, then it is for all others.

Our hypotheses on the collision kernel $K(w,\xi)$ are that it is bounded, and that collisions between particles with large relative velocities are soft. More precisely, there exist constants $K_0,\varepsilon>0$ such that

$$0\leqslant K(w,\xi)\leqslant\frac{K_0 w}{1+w\log^{1+\varepsilon}(1+w)}. \tag{H1}$$

This hypothesis will be required for Theorem 2.3. For Theorem 2.4 we impose an additional small relative velocity cutoff and a lower bound on the collision kernel. Namely, we suppose there exists a constant $R$ such that

$$K(w,\xi)=0\quad\text{for } w<R\quad\text{and}\quad K(w,\xi)\geqslant\beta(\xi)\sup_{|\xi|\leqslant 1}K(w,\xi), \tag{H2}$$

where $\beta(\xi)$ is positive on a set of positive measure. Hypothesis (H2) was also used in [4,5,7].

## 2. Principal results

The general theory of weak (renormalized) solutions of (1) was developed in [9–11]. In the one-dimensional case, for bounded collision kernels $K$, Eq. (1) can be shown to hold in the sense of distributions, and the weak solutions satisfy the global energy conservation [6]. After certain truncation of the collision kernel, the existence of unique weak solutions was obtained in [1]. The approach that we pursue in the present work allows us to obtain solutions with better regularity properties, which are classical solutions if the initial data are sufficiently nice.

**Definition 2.1.** A *strong solution* $f(x,v,t)$ of the Boltzmann equation (1) on a time interval $[0,T)$ is a distributional solution with the properties that $(f \geqslant 0)$ $f \in C([0,T); L^1(\mathbb{T}_x \times \mathbb{R}^3_v)) \cap L^1((0,T); L^\infty(\mathbb{T}_x; L^1(\mathbb{R}^3_v)))$.

Under the condition $\sup K < +\infty$, which follows from (H1), this coincides with the definition used in [11]. A key ingredient of our analysis is to work with $L^\infty_x(L^1_v)$, because control of solutions in this space implies uniqueness, continuous dependence and further propagation properties of solutions. This space has a disadvantage expressed by the fact that the evolution given by the free-streaming flow $S_t : f_0(x,v) \mapsto f_0(x-tv,v)$, $t \geqslant 0$, is not bounded on $L^\infty_x(L^1_v)$. (This disadvantage would be remedied if we were to work with a bounded velocity space.) With this in mind we introduce a slightly more restrictive function class, adapted to the problem at hand.

**Definition 2.2.** For functions $f(x,v)$ on phase space, define the norm

$$\|f\|_X := \sup_{x\in\mathbb{T}_x} \sup_{p\in\mathbb{R}^+} \int_{\mathbb{R}^3_v} \big|(S_p f)(x,v)\big| \,\mathrm{d}v. \tag{6}$$

Denote by $X$ the space of functions $f$ which are independent of $(x_2,x_3)$ and periodic in $x_1$ for which this norm is finite.

A simple criterion for $f \in L^\infty_x(L^1_v)$ to be in $X$ is for there to be a uniform upper envelope $|f(x,v)| \leqslant g(v)$, for some $g \in L^1_v$.

Our approach is based on the Duhamel principle, writing (1) as an integral equation, and remarking that for $f \geqslant 0$ an inequality holds when we drop the $Q^-$ term in the collision operator. Thus, we obtain the following inequality for the evolution of the $X$-norm

$$\|f\|_X(t) \leqslant \|f_0\|_X + \int_0^t \big\|S_{t-s}Q^+\big\|_X(s)\,\mathrm{d}s. \tag{7}$$

A second key ingredient of the result is to use the effect of the angular averaging present in $Q^+$ to estimate the term $\|S_{t-s}Q^+\|_X$. This is achieved by means of the following calculation: for every $q>0$,

$$\int_{\mathbb{R}^3_v} Q^+(x-qv,v)\,\mathrm{d}v \leqslant 2\pi \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_{v_*}} \widetilde{K}\big(|v-v_*|\big) \int_{-1}^{1} F\big(x-q\big(v-|v-v_*|\xi e_1\big), v, v_*\big)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}v_*,$$

where $\widetilde{K}(w) = \sup_{|\xi|\leqslant 1} K(w,\xi)$, $F(x,v,v_*) = f(x,v)f(x,v_*)$, and $e_1$ is the unit vector in the direction of $x_1$. The integration in $\xi$ now plays a role of the averaging in the $x$ variable. By deriving careful estimates of the above expression and using the global bounds of the relative entropy we obtain an upper bound for $\|f(\cdot,t)\|_X$ in terms of the quantity $\varphi(t)$ which satisfies the integral identity

$$\varphi(t) = c_0 + \int_0^t \min\left\{c_1\varphi^2(s), \left(\frac{\alpha}{t-s}+c_2\right)\frac{\varphi(s)}{\log\varphi(s)}\right\}\mathrm{d}s, \tag{8}$$

where $\alpha = C_0K_0H(f_0|M)$ for a universal constant $C_0$. The constants $c_1$ and $c_2$ depend upon $H(f_0|M)$, $K_0$ and $\varepsilon$, while $c_0$ depends upon the initial data. The question of global existence of $\varphi(t)$ in (8) depends solely upon $\alpha$; for $\alpha \leqslant 1$ solutions exist globally in time, whereas for $\alpha > 1$ solutions blow up in finite time. With these facts one obtains the following result:

**Theorem 2.3.** *Consider the Boltzmann equation* (1) *whose collision kernel satisfies* (H1). *Given initial data* $f_0(x,v) \in X$ *such that* $f_0 \geqslant 0$ *and* $H(f_0|M)$ *is finite for a Maxwellian* $M(v)$, *then a strong solution* $f(x,v,t) \in X$ *exists locally in time. If for some Maxwellian* $M_0(v)$ *we have* $\alpha = C_0 K_0 H(f_0|M_0) \leqslant 1$, *then this strong solution exists globally in time, and* $\|\rho(x,t)\|_{L_x^\infty} \leqslant \|f(x,v,t)\|_X \leqslant \varphi(t)$.

The function $\varphi(t)$ is bounded above by $C\exp(\sqrt{t/\beta})$ when $\alpha < 1$, for a constant $\beta = \beta(\alpha)$ which tends to zero as $\alpha$ increases to $\alpha = 1$.

In the case $\alpha > 1$, there are constants $n = n(\alpha)$, $N = N(\alpha)$ such that for initial data $f_0(x,v) \geqslant 0$ in $X$ with $1 < \inf_M(C_0 K_0 H(f_0|M)) = \alpha < \infty$ then a strong solution $f(x,v,t)$ of (1) exists at least over a time interval $0 \leqslant t < T$, where $T = N/\|f_0\|_X^{1/n}$. The constants $n(\alpha)$ and $N(\alpha)$ diverge to infinity in decreasing $\alpha \to 1$, and tend as $\alpha \to \infty$ to $n = 1$ and $N = c_1^{-1}$.

When the stronger hypothesis (H2) is also imposed on the collision kernel $K(w,\xi)$ we have a second main theorem, which estimates $\|f(\cdot,t)\|_X$ in terms of the solution $\varphi(t)$ of the integral identity

$$\varphi(t) = \varphi(0) + \int_0^t \min\left\{c_1\varphi^2(s), \left(\frac{1}{t-s} + 1\right)a(s)\right\} \mathrm{d}s. \tag{9}$$

Here, $a(t) = c_3 \iiiint f(x,v,t) f(x,v_*,t) K(|v-v_*|, \sigma\cdot(v-v_*)/|v-v_*|)|v-v_*|^2 \, \mathrm{d}S_\sigma \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}x$ is related to the Bony functional [3,4,13], which is integrable in time. The constant $c_3$ depends upon $K$. Solutions of (9) cannot blow up in finite time.

**Theorem 2.4.** *Assume that the additional hypothesis* (H2) *is imposed on the Boltzmann collision kernel. Suppose that initial data* $0 \leqslant f_0(x,v) \in X$ *is given such that*

$$A(f_0) + E(f_0) < +\infty. \tag{10}$$

*Then there exists a strong solution* $f(x,v,t)$ *for all* $t \in \mathbb{R}^+$.

Condition (H2) is not necessary for the global existence of weak solutions, as is shown in [6]. However, it plays an important role in our present estimates of strong solutions. Theorem 2.4 does not give information about a rate of growth of $\|f(\cdot,t)\|_X$, and indeed it may be that there does not exist a quantitative upper bound. Solutions may have infinite entropy if their initial data does so, but starting with finite initial entropy then this property is inherited by the solution.

There are a number of results that follow directly from the existence theorems of this section. The solutions above are unique in the class $L^\infty_{\mathrm{loc}}(\mathbb{R}_t^+; X)$. However more than this, the weak/strong uniqueness principle in P.-L. Lions [11] implies that weak solutions produced by other means coincide with our solution. We further show certain propagation properties of moments and derivatives by the Boltzmann equation flow.

**Theorem 2.5.** *Suppose that* $f_0 \in X$ *and that the conditions of either Theorem* 2.3 *or Theorem* 2.4 *are satisfied. If in addition for some nonnegative integers* $k, \ell, m$ *and for all multiindices* $|\kappa| \leqslant k$, $|\lambda| \leqslant \ell$, $|\mu| \leqslant m$ *we have* $|v^\kappa| \partial_v^\lambda \partial_x^\mu f_0 \in L^1(\mathbb{T}_x \times \mathbb{R}_v^3)$, *then*

$$\sum_{|\kappa|\leqslant k,\, |\lambda|\leqslant \ell,\, |\mu|\leqslant m} \int_{\mathbb{T}_x} \int_{\mathbb{R}_v^3} |v^\kappa| \left|\partial_v^\lambda \partial_x^\mu f(x,v,t)\right| \mathrm{d}v \, \mathrm{d}x \leqslant \varphi_{k\ell m}(t) < +\infty,$$

*on the maximal interval of existence of the solution* ($[0,T)$ *in case of Theorem* 2.3 *with* $\alpha > 1$ *and* $[0,+\infty)$ *otherwise*).

The growth rate $\varphi_{k\ell m}(t)$ is bounded by $c_{k\ell m}\exp(C\exp(\sqrt{t/\beta}))$ in the case $\alpha < 1$. Theorem 2.5 implies that if the initial data is sufficiently smooth and decays rapidly at infinity then one-dimensional solutions of (1) with controlled relative entropy are in fact classical solutions.

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