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Mathematical Problems in Mechanics

On Saint Venant's compatibility conditions and Poincaré's lemma

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Abstract

Saint Venant's theorem constitutes a classical characterization of smooth matrix fields as linearized strain tensor fields. This theorem has been extended to matrix fields with components in L^2 by the second author and P. Ciarlet, Jr. in 2005. One objective of this Note is to further extend this characterization to matrix fields whose components are only in H^{-1}. Another objective is to demonstrate that Saint Venant's theorem is in fact nothing but the matrix analog of Poincaré's lemma. ***To cite this article: C. Amrouche et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Sur les conditions de compatibilité de Saint Venant et le lemme de Poincaré. Le théorème de Saint Venant constitue une caractérisation classique de champs de matrices réguliers comme des champs de tenseurs de déformation linéarisés. Ce théorème a été étendu aux champs de matrices avec des composantes dans L^2 par le second auteur et P. Ciarlet, Jr. en 2005. Un objectif de cette Note est d'étendre cette caractérisation aux champs de matrices dont les composantes sont seulement dans H^{-1}. Un autre objectif est de démontrer que le théorème de Saint Venant n'est autre que l'analogue matriciel du lemme de Poincaré. ***Pour citer cet article : C. Amrouche et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Notations and preliminaries

Latin indices vary in the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

Let V denote a normed vector space. The notation V' designates the dual space of V and ${}_{V'}\langle\cdot,\cdot\rangle_V$ denotes the duality bracket between V' and V. Given a subspace W of V, the notation $W^0 := \{v' \in V';\ {}_{V'}\langle v', w\rangle_V = 0$ for all $w \in W\}$ designates the polar set of W.

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Let U and V denote two vector spaces and let $A:U \to V$ be a linear operator. Then $\mathbf{Ker}\, A \subset U$ and $\mathbf{Im}\, A \subset V$ respectively designate the kernel and the image of A.

Let Ω be an open subset of $\mathbb{R}^3$ and let $x = (x_i)$ designate a generic point in Ω. Partial derivative operators of the first, second, and third order are then denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$. The same symbols also designate partial derivatives in the sense of distributions.

Spaces of functions, vector fields, and matrix fields, defined over Ω are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields.

The notations $C^m(\Omega)$, $m \geqslant 0$, and $C^\infty(\Omega)$ denote the usual spaces of continuously differentiable functions; the notation $D(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and have compact supports in Ω. The notation $D'(\Omega)$ denotes the space of distributions defined over Ω. The notations $H^m(\Omega)$, $m \in \mathbb{Z}$, with $H^0(\Omega) = L^2(\Omega)$, and $H_0^1(\Omega)$ designate the usual Sobolev spaces.

The *vector gradient operator* $\mathbf{grad}: D'(\Omega) \to \mathbf{D}'(\Omega)$ is defined by $(\mathbf{grad}\, v)_i := \partial_i v$ for any $v \in D'(\Omega)$. The *divergence operator* $\operatorname{div}: \mathbf{D}'(\Omega) \to D'(\Omega)$ is defined by $\operatorname{div} \mathbf{v} =: \partial_i v_i$ for any $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$. The *vector curl operator* $\mathbf{curl}: \mathbf{D}'(\Omega) \to \mathbf{D}'(\Omega)$ is defined by $(\mathbf{curl}\, \mathbf{v})_i =: \varepsilon_{ijk} \partial_j v_k$ for any $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, where (ε_{ijk}) denotes the orientation tensor. The *matrix gradient operator* $\boldsymbol{\nabla}: \mathbf{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by $(\boldsymbol{\nabla} \mathbf{v})_{ij} := \partial_j v_i$ for any $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$. The *vector divergence operator* $\mathbf{div}: \mathbb{D}'(\Omega) \to \mathbf{D}'(\Omega)$ is defined by $(\mathbf{div}\, \mathbf{e})_i := \partial_j e_{ij}$ for any $\mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$. The *matrix Laplacian* $\Delta: \mathbb{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by $(\Delta \mathbf{e})_{ij} := \Delta e_{ij}$ for any $\mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$. The *matrix curl operator* $\mathbf{CURL}: \mathbb{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{CURL}\, \mathbf{e})_{ij} := \varepsilon_{ilk} \partial_l e_{jk} \quad \text{for any } \mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

For any vector field $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, the symmetric matrix field $\boldsymbol{\nabla}_s \mathbf{v} \in \mathbb{D}'_s(\Omega)$ is defined by

$$\boldsymbol{\nabla}_s \mathbf{v} := \frac{1}{2}\big(\boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{T}} + \boldsymbol{\nabla} \boldsymbol{v}\big),$$

or equivalently, by $(\boldsymbol{\nabla}_s \mathbf{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. When Ω is connected, the kernel of the operator $\boldsymbol{\nabla}_s$ has the well-known characterization

$$\mathbf{Ker}\, \boldsymbol{\nabla}_s = \big\{ \mathbf{v} \in \mathbf{D}'(\Omega);\ \boldsymbol{\nabla}_s \mathbf{v} = \mathbf{0} \text{ in } \mathbb{D}'(\Omega) \big\} = \big\{ \mathbf{v} = \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3,\ \boldsymbol{b} \in \mathbb{R}^3 \big\},$$

where $\mathbf{id}_\Omega$ denotes the identity mapping of the set Ω.

A *domain* in $\mathbb{R}^3$ is a bounded, connected, open subset of $\mathbb{R}^3$ whose boundary is Lipschitz-continuous.

The detailed proofs of the results announced in this Note are given in [2].

2. The operator CURLCURL

Let Ω be an open subset of $\mathbb{R}^3$. For any matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$, the matrix field $\mathbf{CURLCURL}\, \mathbf{e} \in \mathbb{D}'(\Omega)$ is defined by

$$\mathbf{CURLCURL}\, \mathbf{e} := \mathbf{CURL}(\mathbf{CURL}\, \mathbf{e}),$$

or equivalently by $(\mathbf{CURLCURL}\, \mathbf{e})_{ij} := \varepsilon_{ikl} \varepsilon_{jmn} \partial_{ln} e_{km}$.

One objective of this Note is to show that the operator $\mathbf{CURLCURL}: \mathbb{D}'_s(\Omega) \to \mathbb{D}'(\Omega)$ defined in this fashion is the 'matrix analog' of the 'vector' operator $\mathbf{curl}: \mathbf{D}'(\Omega) \to \mathbf{D}'(\Omega)$. The next theorem, which lists some algebraic properties of this operator, includes some identities that constitute a first contribution to this objective.

Theorem 2.1. *Let Ω be any open subset of $\mathbb{R}^3$. The operator* **CURLCURL** *possesses the following properties*:

(a) *For any matrix field $\mathbf{e} \in \mathbb{D}'_s(\Omega)$,*

$$\mathbf{CURLCURL}\, \mathbf{e} = (\mathbf{CURLCURL}\, \mathbf{e})^{\mathrm{T}} \quad \text{in } \mathbb{D}'(\Omega),$$

$$\mathbf{div}(\mathbf{CURLCURL}\, \mathbf{e}) = \mathbf{0} \quad \text{in } \mathbf{D}'(\Omega),$$

$$\operatorname{tr}(\mathbf{CURLCURL}\, \mathbf{e}) = \Delta(\operatorname{tr} \mathbf{e}) - \operatorname{div}(\mathbf{div}\, \mathbf{e}) \quad \text{in } \mathrm{D}'(\Omega).$$

(b) *Given any matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'_s(\Omega)$, let*

$$R_{ijkl}(\mathbf{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} \quad \text{in } \mathrm{D}'(\Omega).$$

Then each distribution $R_{ijkl}(\mathbf{e})$ that does not identically vanish is equal to some distribution $(\mathbf{CURLCURL\,e})_{pq}$ for appropriate indices p and q, and conversely. Consequently, the eighty-one relations $R_{ijkl}(\mathbf{e}) = 0$ in $\mathrm{D}'(\Omega)$ are equivalent to the six relations $(\mathbf{CURLCURL\,e})_{mn} = 0$ in $\mathrm{D}'(\Omega)$, $m \leqslant n$, i.e., to $\mathbf{CURLCURL\,e} = \mathbf{0}$ in $\mathbb{D}'_s(\Omega)$.

(c) *For any vector field $\mathbf{v} \in \mathbf{D}'(\Omega)$, $\mathbf{CURLCURL}(\nabla_s \mathbf{v}) = \mathbf{0}$ in $\mathbb{D}'(\Omega)$.*

Sketch of proof. The relations of (a) and (c) are established by direct computations.

Let a matrix field $\mathbf{e} = (e_{ij}) \in \mathbb{D}'_s(\Omega)$ be given and let $\boldsymbol{q} = (q_{ij}) := \mathbf{CURLCURL\,e}$. Then a direct computation shows that $q_{11} = R_{2323}(\mathbf{e}), q_{12} = R_{2331}(\mathbf{e}), q_{13} = R_{1223}(\mathbf{e}), q_{22} = R_{1313}(\mathbf{e}), q_{23} = R_{1312}(\mathbf{e}), q_{33} = R_{1212}(\mathbf{e})$. Taking also into account the relations $R_{ijkl}(\mathbf{e}) = 0$ if $i = j$ or $k = l$, $R_{ijkl}(\mathbf{e}) = R_{klij}(\mathbf{e}) = -R_{jikl}(\mathbf{e}) = -R_{ijlk}(\mathbf{e})$, we thus conclude that all the distributions $R_{ijkl}(\mathbf{e})$ that do not identically vanish are known if and only if the six ones appearing above (i.e., $R_{2323}(\mathbf{e}), \ldots, R_{1212}(\mathbf{e})$) are known. This proves (b). □

Note that the relations $\mathbf{div}(\mathbf{CURLCURL\,e}) = \mathbf{0}$ and $\mathbf{CURLCURL}(\nabla_s \mathbf{v}) = \mathbf{0}$, which hold for arbitrary matrix fields $\mathbf{e} \in \mathbb{D}'_s(\Omega)$ and vector fields $\mathbf{v} \in \mathbf{D}'(\Omega)$, are indeed the 'matrix analogs' of the well-known relations $\mathrm{div}(\mathbf{curl\,v}) = 0$ and $\mathbf{curl}(\mathbf{grad}\, v) = \mathbf{0}$, which hold for arbitrary vector fields $\mathbf{v} \in \mathbf{D}'(\Omega)$ and distributions $v \in \mathrm{D}'(\Omega)$.

3. An extension of Saint Venant's theorem

The following $\mathbb{H}^m_s(\Omega)$-*matrix version of J.-L. Lions' lemma*, announced in [1] and proved in [2], plays a key role in the sequel.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^3$ and let a vector field $\mathbf{v} \in \mathbf{D}'(\Omega)$ be such that $\nabla_s \mathbf{v} \in \mathbb{H}^m_s(\Omega)$ for some integer $m \in \mathbb{Z}$. Then $\mathbf{v} \in \mathbf{H}^{m+1}(\Omega)$.*

Let Ω be any open subset in $\mathbb{R}^3$. Given any vector field $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, Theorem 2.1 shows that $\mathbf{CURLCURL}(\nabla_s \mathbf{v}) = \mathbf{0}$ in $\mathbb{D}'_s(\Omega)$, or equivalently, that the *Saint Venant compatibility conditions* $R_{ijkl}(\nabla_s \mathbf{v}) = 0$ hold in $\mathrm{D}'(\Omega)$. It has been known for a long time that the following converse, known as *Saint Venant's theorem*, holds for smooth enough matrix fields: *Let Ω be a simply-connected open subset of $\mathbb{R}^3$. Assume that, for some integer $m \geqslant 2$, a matrix field $\mathbf{e} \in \mathbb{C}^m_s(\Omega)$ satisfies the relations $R_{ijkl}(\mathbf{e}) = 0$ in Ω. Then there exists a vector field $\mathbf{v} \in \mathbf{C}^{m+1}(\Omega)$ such that* $\mathbf{e} = \nabla_s \mathbf{v}$ in Ω (although this result was announced by A.J.C.B. de Saint Venant in 1864, it was not until 1886 that E. Beltrami provided a rigorous proof).

We now show that the same Saint Venant compatibility conditions $R_{ijkl}(\mathbf{e}) = 0$ remain sufficient in a much weaker sense, according to the following *Saint Venant's theorem in $\mathbb{H}^{-1}_s(\Omega)$*.

Theorem 3.2. *Let Ω be a simply-connected domain in $\mathbb{R}^3$ and let $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$ be a matrix field that satisfies $\mathbf{CURLCURL\,e} = \mathbf{0}$ in $\mathbb{H}^{-3}_s(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ that satisfies $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{H}^{-1}_s(\Omega)$.*

All other vector fields $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ satisfying $\mathbf{e} = \nabla_s \tilde{\mathbf{v}}$ in $\mathbb{H}^{-1}_s(\Omega)$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega$ for some vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

Sketch of proof. One can show (see [1,2]) that $-\mathbf{div}: \mathbb{H}^1_{0,s}(\Omega) \to \dot{\mathbf{L}}^2(\Omega) = \mathbf{L}^2(\Omega)/\mathbf{Ker}\,\nabla_s$ is the dual operator of $\nabla_s : \dot{\mathbf{L}}^2(\Omega) \to \mathbb{H}^{-1}_s(\Omega)$ and that the operator $\nabla_s : \dot{\mathbf{L}}^2(\Omega) \to \mathbf{Im}\,\nabla_s = \mathbb{V}^0$, where $\mathbb{V} := \mathbf{Ker}(-\mathbf{div}) \subset \mathbb{H}^1_{0,s}(\Omega)$, is an isomorphism. Consequently, the operator $-\mathbf{div}: (\mathbb{V}^0)' \to \dot{\mathbf{L}}^2(\Omega)$ is also an isomorphism. Besides, the inclusion $\mathbb{V}^0 \subset \mathbb{H}^{-1}_s(\Omega) = (\mathbb{H}^1_{0,s}(\Omega))'$ implies that $(\mathbb{V}^0)'$ can be identified with a (closed) subspace of $\mathbb{H}^1_{0,s}(\Omega)$. We thus reach two conclusions. First, *given any element $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$, there exists a unique matrix field $\mathbf{q}(\dot{\mathbf{v}}) \in (\mathbb{V}^0)' \subset \mathbb{H}^1_{0,s}(\Omega)$ such that* $-\mathbf{div}\,\mathbf{q}(\dot{\mathbf{v}}) = \dot{\mathbf{v}}$ in $\dot{\mathbf{L}}^2(\Omega)$. Second, *there exists a constant $\beta > 0$ such that $\beta\|\mathbf{q}(\dot{\mathbf{v}})\|_{1,\Omega} \leqslant \|\dot{\mathbf{v}}\|_{0,\Omega}$* for all $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$.

(ii) Define two bilinear forms $a : \mathbb{H}^1_{0,s}(\Omega) \times \mathbb{H}^1_{0,s}(\Omega) \to \mathbb{R}$ and $b : \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}^1_{0,s}(\Omega) \to \mathbb{R}$ by

$$a(\mathbf{q}, \mathbf{r}) := \int_\Omega \partial_k q_{ij} \partial_k r_{ij}\, \mathrm{d}x \quad \text{for all } (\mathbf{q}, \mathbf{r}) = \big((q_{ij}), (r_{ij})\big) \in \mathbb{H}^1_{0,s}(\Omega) \times \mathbb{H}^1_{0,s}(\Omega),$$

$$b(\dot{\mathbf{v}}, \mathbf{q}) := -\int_\Omega v_i \partial_j q_{ij}\, \mathrm{d}x \quad \text{for all } (\dot{\mathbf{v}}, \mathbf{q}) = \big((\dot{v}_i), (q_{ij})\big) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}^1_{0,s}(\Omega)$$

(the bilinear form b is indeed unambiguously defined, because $\mathbf{q}$ is symmetric). Clearly, the two bilinear forms are continuous and the bilinear form a is $\mathbb{H}^1_{0,s}(\Omega)$-elliptic. Furthermore, part (i) implies that the bilinear form b satisfies the *Babuška–Brezzi inf–sup condition*. Consequently, *given any element* $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$*, there exists a unique solution* $(\dot{\mathbf{u}}, \mathbf{q}) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}^1_{0,s}(\Omega)$ *to the equations*

$$a(\mathbf{q}, \mathbf{r}) + b(\dot{\mathbf{u}}, \mathbf{r}) = {}_{\mathbb{H}^{-1}_s(\Omega)}\langle \mathbf{e}, \mathbf{r}\rangle_{\mathbb{H}^1_{0,s}(\Omega)} \quad \text{for all } \mathbf{r} \in \mathbb{H}^1_{0,s}(\Omega), \quad \text{and} \quad b(\dot{\mathbf{v}}, \mathbf{q}) = 0 \quad \text{for all } \dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega),$$

or equivalently, to the equations

$$-\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} = \mathbf{e} \quad \text{in } \mathbb{H}^{-1}_s(\Omega) \quad \text{and} \quad \mathbf{div}\, \mathbf{q} = \mathbf{0} \quad \text{in } \dot{\mathbf{L}}^2(\Omega).$$

(iii) Assume that the element $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$ appearing in the right-hand side of the penultimate equation satisfies *in addition* $\mathbf{CURL\,CURL}\,\mathbf{e} = \mathbf{0}$ in $\mathbb{H}^{-3}_s(\Omega)$, so that, by Theorem 2.1(c),

$$\Delta(\mathbf{CURL\,CURL}\,\mathbf{q}) = \mathbf{CURL\,CURL}(\Delta \mathbf{q}) = \mathbf{CURL\,CURL}(\nabla_s \dot{\mathbf{u}} - \mathbf{e}) = \mathbf{0} \quad \text{in } \mathbb{H}^{-3}_s(\Omega).$$

The *hypoellipticity of the Laplacian* (see, e.g., Dautray and Lions [4, Section 2 in Chapter 5]) then implies that $\mathbf{CURL\,CURL}\,\mathbf{q} \in \mathbb{C}^\infty_s(\Omega)$, and Theorem 2.1(a) in turn shows that

$$\Delta(\operatorname{tr} \mathbf{q}) = \operatorname{div}(\mathbf{div}\, \mathbf{q}) + \operatorname{tr}(\mathbf{CURL\,CURL}\,\mathbf{q}) = \operatorname{tr}(\mathbf{CURL\,CURL}\,\mathbf{q}) \in \mathbb{C}^\infty(\Omega).$$

Hence $\operatorname{tr} \mathbf{q} \in C^\infty(\Omega)$, again by the hypoellipticity of the Laplacian.

Using Theorem 2.1(b), we next infer that, for all indices i and k, $R_{ilkl}(\mathbf{q}) = \{\Delta q_{ik} + \partial_{ik}(\operatorname{tr} \mathbf{q})\} \in C^\infty(\Omega)$, which implies that $\Delta \mathbf{q} \in \mathbb{C}^\infty_s(\Omega)$.

Hence, *if* $\mathbf{CURL\,CURL}\,\mathbf{e} = \mathbf{0}$ *in* $\mathbb{H}^{-3}_s(\Omega)$*, the second argument* $\mathbf{q}$ *of the solution* $(\dot{\mathbf{u}}, \mathbf{q}) \in \dot{\mathbf{L}}^2(\Omega) \times \mathbb{H}^1_{0,s}(\Omega)$ *to the equations* $-\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} = \mathbf{e}$ *in* $\mathbb{H}^{-1}_s(\Omega)$ *and* $\mathbf{div}\, \mathbf{q} = \mathbf{0}$ *in* $\dot{\mathbf{L}}^2(\Omega)$ *satisfies*

$$\Delta \mathbf{q} \in \mathbb{C}^\infty_s(\Omega) \quad \text{and} \quad \mathbf{CURL\,CURL}(\Delta \mathbf{q}) = \mathbf{0} \quad \text{in } \Omega.$$

(iv) Since the matrix field $\Delta \mathbf{q} \in \mathbb{C}^\infty_s(\Omega)$ satisfies $\mathbf{CURL\,CURL}(\Delta \mathbf{q}) = \mathbf{0}$ in the simply-connected open set Ω, the *classical Saint Venant theorem* shows that there exists a vector field $\mathbf{w} \in \mathbf{C}^\infty(\Omega)$ such that $\Delta \mathbf{q} = \nabla_s \mathbf{w}$ in Ω (this is the only place where the simple-connectedness of Ω is used).

The vector field $\mathbf{w} \in \mathbf{C}^\infty(\Omega) \subset \mathbf{D}'(\Omega)$ therefore satisfies $\nabla_s \mathbf{w} = \{\nabla_s \dot{\mathbf{u}} - \mathbf{e}\} \in \mathbb{H}^{-1}_s(\Omega)$. Consequently, the $\mathbb{H}^{-1}_s(\Omega)$-*matrix version of J.-L. Lions' lemma* (Theorem 3.1) shows that $\mathbf{w} \in \mathbf{L}^2(\Omega)$. Hence $\dot{\mathbf{v}} := \{\dot{\mathbf{u}} - \dot{\mathbf{w}}\} \in \dot{\mathbf{L}}^2(\Omega)$ satisfies $\mathbf{e} = \nabla_s \dot{\mathbf{v}}$ in $\mathbb{H}^{-1}_s(\Omega)$, which concludes the existence proof.

That all other solutions $\tilde{\mathbf{v}}$ of the equation $\mathbf{e} = \nabla_s \tilde{\mathbf{v}}$ are of the form indicated above follows from the characterization of the space $\mathbf{Ker}\, \nabla_s$ recalled earlier. □

Note that *the equations* (encountered in part (ii) of the above proof) $-\Delta \mathbf{q} + \nabla_s \dot{\mathbf{u}} = \mathbf{e}$ in $\mathbb{H}^{-1}_s(\Omega)$ and $\mathbf{div}\, \mathbf{q} = \mathbf{0}$ in $\dot{\mathbf{L}}^2(\Omega)$ *constitute the 'matrix analog' of the familiar stationary Stokes problem*. We recall that this problem consists in finding a pair $(\dot{p}, \mathbf{u}) \in \dot{L}^2(\Omega) \times \mathbf{H}^1_0(\Omega)$, where $\dot{L}^2(\Omega) := L^2(\Omega)/\mathbb{R}$, that satisfies the equations $-\nu \Delta \mathbf{u} + \mathbf{grad}\, \dot{p} = \mathbf{f}$ in $\mathbf{H}^{-1}(\Omega)$ and $\operatorname{div} \mathbf{u} = \mathbf{0}$ in $\dot{L}^2(\Omega)$. This observation explains in particular why the existence theory used in part (ii) resembles that used for the Stokes problem (see Girault and Raviart [7, Section 5.1]).

As expected, a *Saint Venant's theorem in* $\mathbb{L}^2_s(\Omega)$, i.e., similar to that of Theorem 3.2 but with a 'shift by +1' in the regularities of both fields $\mathbf{e}$ and $\mathbf{v}$, likewise holds:

Theorem 3.3. *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$ *and let* $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$ *be a matrix field that satisfies* $\mathbf{CURL\,CURL}\,\mathbf{e} = \mathbf{0}$ *in* $\mathbb{H}^{-2}_s(\Omega)$*. Then there exists a vector field* $\mathbf{v} \in \mathbf{H}^1(\Omega)$ *that satisfies* $\mathbf{e} = \nabla_s \mathbf{v}$ *in* $\mathbb{L}^2_s(\Omega)$*.*

Proof. Since $\mathbb{L}^2_s(\Omega) \subset \mathbb{H}^{-1}_s(\Omega)$, Theorem 3.2 shows that there exists $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}^2_s(\Omega)$. Theorem 3.1 then implies that $\mathbf{v} \in \mathbf{H}^1(\Omega)$. □

Saint Venant's theorem in $\mathbb{L}^2_s(\Omega)$ is due to Ciarlet and Ciarlet, Jr. [3]. More recently, another proof of this result was given by Geymonat and Krasucki [5]. See also Geymonat and Krasucki [6], who showed how Saint Venant's theorem in $\mathbb{L}^2_s(\Omega)$ can be extended to domains Ω that are not simply-connected by means of *Beltrami's functions*.

In Ciarlet and Ciarlet, Jr. [3], it is also shown how Saint Venant's theorem in $\mathbb{L}^2_s(\Omega)$ can be put to use so as to provide another reformulation of the pure traction problem of linearized three-dimensional elasticity posed over simply-connected domains, where *the linearized strains* (and consequently also the stresses since the constitutive equation is invertible) *are the primary unknowns*.

4. Saint Venant's theorem and Poincaré's lemma

First, we emphasize that the *Saint Venant theorem in* $\mathbb{H}^{-1}_s(\Omega)$ (*Theorem* 3.2) *constitutes the matrix analog of the Poincaré lemma in* $\mathbf{H}^{-1}(\Omega)$, which takes the following form: *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$. *If a vector field* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ *satisfies* $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ *in* $\mathbf{H}^{-2}(\Omega)$, *then there exists a function* $p \in L^2(\Omega)$ *such that* $\mathbf{h} = \mathbf{grad}\, p$ (Poincaré's lemma in $\mathbf{H}^{-1}(\Omega)$, which is due to Ciarlet and Ciarlet, Jr. [3], was later given a different and simpler proof by Kesavan [8]). In other words, the 'vector' operators **curl** and **grad** appearing in Poincaré's lemma are 'replaced' in Theorem 3.2 by their 'matrix analogs' **CURLCURL** and ∇_s.

Second, we record the following equivalence, which is due to Kesavan [8]: *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$. *Then the following statements are equivalent*:

(a) *If* $v \in D'(\Omega)$ *is such that* $\mathbf{grad}\, v \in \mathbf{H}^{-1}(\Omega)$, *then* $v \in L^2(\Omega)$.

(b) *If* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ *satisfies* $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ *in* $\mathbf{H}^{-2}(\Omega)$, *then* $\mathbf{h} = \mathbf{grad}\, p$ *for some* $p \in L^2(\Omega)$.

In other words, *J.-L. Lions' lemma in* $\mathbf{H}^{-1}(\Omega)$ (statement (a)) *is equivalent to Poincaré's lemma in* $\mathbf{H}^{-1}(\Omega)$ (statement (b)).

We now show that, likewise, the $\mathbb{H}^{-1}_s(\Omega)$-*matrix version of J.-L. Lions' lemma* (Theorem 3.1; statement (a) in the next theorem) *is equivalent to Saint Venant's theorem in* $\mathbb{H}^{-1}_s(\Omega)$ (Theorem 3.2; statement (b) in the next theorem):

Theorem 4.1. *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$. *The following statements are equivalent*:

(a) *If* $\mathbf{w} \in \mathbf{D}'(\Omega)$ *satisfies* $\nabla_s \mathbf{w} \in \mathbb{H}^{-1}_s(\Omega)$, *then* $\mathbf{w} \in \mathbf{L}^2(\Omega)$.

(b) *If* $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$ *satisfies* $\mathbf{CURLCURL}\,\mathbf{e} = \mathbf{0}$ *in* $\mathbb{H}^{-3}_s(\Omega)$, *then* $\mathbf{e} = \nabla_s \mathbf{v}$ *for some* $\mathbf{v} \in \mathbf{L}^2(\Omega)$.

Proof. Theorem 3.1 is used in part (iv) of the proof of Theorem 3.2. Hence (a) implies (b).

Assume next that (b) holds and let $\mathbf{w} \in \mathbf{D}'(\Omega)$ be such that $\nabla_s \mathbf{w} \in \mathbb{H}^{-1}_s(\Omega)$. Noting that $\mathbf{CURLCURL}(\nabla_s \mathbf{w}) = \mathbf{0}$ by Theorem 3.1(c), we infer from (b) that $\nabla_s \mathbf{w} = \nabla_s \mathbf{v}$ for some $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Hence $(\mathbf{w} - \mathbf{v}) \in \mathbf{Ker}\,\nabla_s \subset \mathbf{L}^2(\Omega)$ and thus $\mathbf{w} \in \mathbf{L}^2(\Omega)$. Hence (b) implies (a). □

Theorem 4.1 constitutes another evidence that *Saint Venant theorem in* $\mathbb{H}^{-1}_s(\Omega)$ *is indeed the matrix analog of Poincaré's lemma in* $\mathbf{H}^{-1}(\Omega)$.

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