

Partial Differential Equations

# Critical space for the parabolic-parabolic Keller–Segel model in $\mathbb{R}^d$

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## Abstract

We study the Keller–Segel system in  $\mathbb{R}^d$  when the chemoattractant concentration is described by a parabolic equation. We prove that the critical space, with some similarity to the elliptic case, is that the initial bacteria density satisfies  $n_0 \in L^a(\mathbb{R}^d)$ ,  $a > d/2$ , and that the chemoattractant concentration satisfies  $\nabla c_0 \in L^d(\mathbb{R}^d)$ . In these spaces, we prove that small initial data give rise to global solutions that vanish as the heat equation for large times and that exhibit a regularizing effect of hypercontractivity type. **To cite this article:** L. Corrias, B. Perthame, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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## Résumé

**Espace critique pour le modèle de Keller–Segel parabolique-parabolique dans  $\mathbb{R}^d$ .** Nous considérons le système de Keller–Segel posé sur  $\mathbb{R}^d$  dans le cas d'une équation parabolique sur le chemoattractant. Nous démontrons que l'espace critique, comme dans le cas elliptique, est que la densité bactérienne initiale vérifie  $n_0 \in L^a(\mathbb{R}^d)$ ,  $a > d/2$ , et que la concentration initiale de chemoattractant vérifie  $\nabla c_0 \in L^d(\mathbb{R}^d)$ . Dans ces espaces, une donnée initiale petite donne des solutions globales qui tendent vers 0 en temps grand comme l'équation de la chaleur ainsi que des effets régularisants de type hypercontractifs. **Pour citer cet article :** L. Corrias, B. Perthame, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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## Version française abrégée

Le système de Keller–Segel [11] est célèbre en biologie car il sert à décrire le mouvement collectif de bactéries, menant éventuellement à l'aggrégation des cellules par effet chémo-tactique [12,9,8]. Plusieurs versions ont été étudiées et la plus simple postule une équation elliptique pour le chémoattractant. Ici nous considérons une équation parabolique à la fois pour la densité cellulaire  $n$  et la concentration de chémoattractant  $c$

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c), & x \in \mathbb{R}^d, t > 0, \\ \frac{\partial c}{\partial t} - \Delta c = n - c, & x \in \mathbb{R}^d, t > 0, \end{cases} \quad (1)$$

avec des données initiales  $n_0 \geq 0$ ,  $c_0 \geq 0$  et on suppose que le système est posé dans l'espace entier pour simplifier.

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Le cas où le chemoattractant est décrit par l'équation elliptique  $-\Delta c = n$  est assez bien compris depuis les premiers travaux de [10,15]. En dimension  $d = 2$ , l'espace critique est  $L^1(\mathbb{R}^d)$  : en-dessous d'une masse critique  $M = 8\pi/\chi$ , le système admet des solutions globales, au-dessus les solutions explosent en temps fini (voir [6,2,13,1]). Ce type de résultat peut se généraliser en toute dimension quand le chemoattractant est obtenu par la convolution avec  $\log|x|$  (comme le potentiel bidimensionnel), voir [4]. Les singularités, au-delà de la masse critique, sont des concentrations étudiées par exemple dans [7,17]. En dimension  $d > 2$  les résultats sont plus faibles et la masse ne joue plus le rôle central. Il a été démontré dans [5] que l'espace critique est  $L^{d/2}(\mathbb{R}^d)$  : pour une norme  $\|n_0\|_{L^{d/2}(\mathbb{R}^d)}$  assez petite les solutions sont globales. Par contre le critère d'explosion est moins précis et utilise une norme, ayant la même homogénéité que  $L^{d/2}(\mathbb{R}^d)$ , mais plus forte. Le type de singularité est aussi plus varié qu'en deux dimensions, [3].

Dans cette Note, nous étendons ce premier résultat au cas du système parabolique-parabolique (1) et améliorons donc des résultats antérieurs ([14]) en travaillant dans les espaces suivant

$$n_0 \in (L^1 \cap L^a)(\mathbb{R}^d), \quad \frac{d}{2} < a \leq d, \quad \nabla c_0 \in L^d(\mathbb{R}^d), \quad (2)$$

où  $a$  est arbitrairement proche de  $\frac{d}{2}$ .

**Théorème 1.** *Sous l'hypothèse (2) et pour  $d \geq 3$ , il existe une constante  $C(d, a)$  telle que si*

$$\|n_0\|_{L^a(\mathbb{R}^d)} + \|\nabla c_0\|_{L^d(\mathbb{R}^d)} \leq C(d, a),$$

alors le système (1) admet une solution positive faible et globale qui vérifie pour tout  $\epsilon > 0$

$$\|n(t) - G(t) \star n_0\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{\frac{1}{2}-d+\epsilon}, \quad t \rightarrow \infty, \quad (3)$$

où  $G$  est le noyau de la chaleur.

Comme dans [5] on peut démontrer l'explosion pour des données initiales assez grandes sous certaines conditions qui sont toutefois trop éloignées des normes en question ici et nous n'énonçons donc pas le résultat. Notons également que, contrairement au cas parabolique-elliptique, le cas  $a = d/2$  ne semble pas accessible directement.

La preuve est donnée dans la version anglaise et utilise un argument d'injection de Sobolev. Les méthodes d'énergie, plus précises (voir [6,2]), ne semblent pas adaptées aux espaces concernés en dimension  $d > 2$ .

## 1. Introduction

The Keller–Segel system [11] is famous in biology for describing the collective motion of cells (bacteria for instance) under chemotactic attraction, leading possibly to aggregation of cells (chemotactic collapse), see [12,9,8]. Several versions of the model have been studied and the simplest uses an elliptic equation for the chemoattractant. Here we consider the case of a parabolic equation both for the cell density  $n(t, x)$  and the chemoattractant concentration  $c(t, x)$ , i.e. we deal with the system

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c), & x \in \mathbb{R}^d, t > 0, \\ \frac{\partial c}{\partial t} - \Delta c = n - c, & x \in \mathbb{R}^d, t > 0, \end{cases} \quad (1)$$

that we set in the whole space for simplicity and with initial data denoted by  $n_0 \geq 0, c_0 \geq 0$ . Besides that they are nonnegative, the solutions satisfy mass conservation

$$\int_{\mathbb{R}^d} n(t, x) dx = \int_{\mathbb{R}^d} n_0(x) dx := M. \quad (2)$$

When the chemoattractant is described by the elliptic equation  $\Delta c = -n$  the theory is already more advanced since the pioneering works of [10,15]. In dimension  $d = 2$ ,  $L^1(\mathbb{R}^d)$  is the critical space: for  $M$  below the critical mass  $8\pi/\chi$  there are global weak solutions, for  $M$  above the critical mass there is blow-up in finite time [6,2,13,1]. This kind of results can be extended to all dimensions with a *log* potential (as in two dimensions), see [4]. The singularities are Dirac concentrations that have been studied for instance in [7,17]. In dimension  $d > 2$ , it has been proved in [5] that  $L^{d/2}(\mathbb{R}^d)$  is the critical space: for small norms  $\|n_0\|_{L^{d/2}(\mathbb{R}^d)}$  there are global weak solutions, but the blow-up criteria

uses a stronger norm with the same homogeneity. Also the singularities that may occur are much more numerous than in two dimensions, [3].

Here we are interested in the parabolic system (1) and we improve previous results (see [14] for instance) in working in more precise spaces given by

$$n_0 \in (L^1 \cap L^a)(\mathbb{R}^d), \quad \frac{d}{2} < a \leq d, \quad \nabla c_0 \in L^d(\mathbb{R}^d), \tag{3}$$

where  $a$  is arbitrarily close to  $\frac{d}{2}$ .

**Theorem 1.1.** *Under assumption (3) and for  $d \geq 3$ , there is a constant  $C(d, a)$  such that, for*

$$\|n_0\|_{L^a(\mathbb{R}^d)} + \|\nabla c_0\|_{L^d(\mathbb{R}^d)} \leq C(d, a), \tag{4}$$

the system (1) has at least one weak and global positive solution which satisfies the hypercontractivity type estimate for any  $\epsilon > 0$

$$\|n(t) - G(t) \star n_0\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{\frac{1}{2}-d+\epsilon}, \quad t \rightarrow \infty, \tag{5}$$

where  $G$  denotes the heat kernel.

In other terms, in this regime, the quadratic nonlinearity does not act in long time and the solution  $n$  behaves as the homogeneous heat equation with initial data  $n_0$ . Let us note that the critical case  $a = d/2$  is not accessible by the method below. The best known result in bounded domains is  $a = d/2 + 1$ , [16].

## 2. Existence proof

We will consider for simplicity and without loss of generality  $\chi = 1$ . For the existence part, it is enough to obtain a priori estimates. Then it is in the folklore of the subject to prove them rigorously for smoother potentials and pass to the limit. For solutions of (1), we denote by  $G$  the fundamental solution of the heat equation and  $G \star n$  is the  $x$  convolution of  $G$  and  $n$ . Then, we write

$$n(t, x) = G(t) \star n_0 - \int_0^t \nabla G(t-s) \star (n(s) \nabla c(s)) \, ds, \tag{6}$$

$$c(t, x) = e^{-t} G(t) \star c_0 + \int_0^t e^{(s-t)} G(t-s) \star n(s) \, ds. \tag{7}$$

In order to estimate the  $L^p$  norm of  $n$ , we depart from the identity

$$\frac{d}{dt} \int_{\mathbb{R}^d} n^p \, dx = -4 \frac{p-1}{p} \int_{\mathbb{R}^d} |\nabla n^{\frac{p}{2}}|^2 \, dx + 2(p-1) \int_{\mathbb{R}^d} n^{\frac{p}{2}} \nabla n^{\frac{p}{2}} \cdot \nabla c \, dx. \tag{8}$$

The last term is a priori the bad one and we control it as

$$\begin{aligned} \int_{\mathbb{R}^d} n^{\frac{p}{2}} \nabla n^{\frac{p}{2}} \cdot \nabla c \, dx &\leq e^{-t} \int_{\mathbb{R}^d} n^{\frac{p}{2}} |\nabla n^{\frac{p}{2}}| |\nabla G(t) \star c_0| \, dx + \int_{\mathbb{R}^d} n^{\frac{p}{2}} |\nabla n^{\frac{p}{2}}| \int_0^t e^{(s-t)} |\nabla G(t-s) \star n(s)| \, dx \\ &\leq e^{-t} \|n^{\frac{p}{2}}\|_{r_1} \|\nabla n^{\frac{p}{2}}\|_{r_2} \|G(t) \star \nabla c_0\|_{r_3} + \|n^{\frac{p}{2}}\|_{q_1} \|\nabla n^{\frac{p}{2}}\|_{q_2} \int_0^t e^{(s-t)} \|\nabla G(t-s) \star n(s)\|_{q_3} \, ds \\ &\leq e^{-t} \|n^{\frac{p}{2}}\|_{r_1} \|\nabla n^{\frac{p}{2}}\|_{r_2} \|\nabla c_0\|_{r_3} \\ &\quad + C \|n^{\frac{p}{2}}\|_{q_1} \|\nabla n^{\frac{p}{2}}\|_{q_2} \int_0^t e^{(s-t)} \frac{1}{\sqrt{t-s}^{d(\frac{1}{q}-\frac{1}{q_3})+1}} \|n(s)\|_{q_3} \, ds = I_1 + I_2, \end{aligned} \tag{9}$$

where

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad q \leq q_3.$$

Concerning  $I_1$ , we choose  $r_1 = \frac{2d}{d-2}$ ,  $r_2 = 2$  and  $r_3 = d$ . Then, using the Sobolev injection  $\|n^{\frac{p}{2}}\|_{r_1} \leq C \|\nabla n^{\frac{p}{2}}\|_{r_2}$ , we arrive at

$$I_1 \leq C e^{-t} \|\nabla n^{\frac{p}{2}}\|_2^2 \|\nabla c_0\|_d.$$

Next, we estimate  $I_2$  using again  $q_2 = 2$  and  $q_1 = \frac{2d}{d-2}$  and thus  $q_3 = d$ . As before, we find

$$\begin{aligned} I_2 &\leq C \|\nabla n^{\frac{p}{2}}\|_2^2 \left( \sup_{0 \leq s \leq t} \|n(s)\|_q \right) \int_0^t e^{(s-t)} \frac{1}{\sqrt{t-s}^{d(\frac{1}{q}-\frac{1}{d})+1}} ds \\ &\leq C \|\nabla n^{\frac{p}{2}}\|_2^2 \left( \sup_{0 \leq s \leq t} \|n(s)\|_q \right) \Gamma(\mathcal{B}), \end{aligned}$$

with  $\Gamma(\mathcal{B}) < \infty$  iff  $\mathcal{B} = \frac{1}{2} - \frac{d}{2} \left( \frac{1}{q} - \frac{1}{d} \right) = 1 - \frac{d}{2q} > 0$ . This enforces the choice  $q > \frac{d}{2}$ .

Finally, we have proved that for any  $p > 1$  and  $\frac{d}{2} < q \leq d$ , we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} n^p \leq 2(p-1) \|\nabla n^{\frac{p}{2}}\|_2^2 \left[ C e^{-t} \|\nabla c_0\|_d + C \Gamma \left( 1 - \frac{d}{2q} \right) \left( \sup_{0 \leq s \leq t} \|n(s)\|_q \right) - \frac{2}{p} \right], \quad t > 0, \quad (10)$$

where the constants are independent of  $p$  and  $q$ . We conclude choosing  $p = q = a$  as in the theorem. If the initial data are such that  $\|n_0\|_a + \|\nabla c_0\|_d$  is small enough compared to  $\frac{1}{a}$ , then the bracket in (10) is negative and thus  $\int n(t, x)^a dx$  decreases and this remains true for all times.

### 3. Long time asymptotic in $L^a(\mathbb{R}^d)$

Going forward in the inequality (10), we can obtain the time decay of  $\|n(t)\|_{L^p(\mathbb{R}^d)}$ , for any  $1 < p \leq a$  with  $a$  given in assumption (3). Indeed, using standard interpolation and the Gagliardo–Nirenberg and Sobolev inequalities, one can obtain for any  $1 \leq p < \infty$ , the inequality

$$\|n\|_{L^p(\mathbb{R}^d)} \leq C(d) \|n\|_{L^1(\mathbb{R}^d)}^{1-\theta} \|\nabla n^{p/2}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta}{p}}, \quad (11)$$

where  $\theta = \frac{pd-d}{pd+2-d} \in [0, 1]$ . If the initial data are such that  $\|n_0\|_a + \|\nabla c_0\|_d$  is small enough compared to  $\frac{1}{a}$  to have the bracket in (10) strictly negative, using the mass conservation (2) and (11) with  $p = a$ , the inequality (10) becomes

$$\frac{d}{dt} \int_{\mathbb{R}^d} n^a(x, t) dx \leq -C \left( \int_{\mathbb{R}^d} n^a(x, t) dx \right)^{1/\theta}. \quad (12)$$

This shows that  $\|n(t)\|_a$  has the same time decay as the homogeneous heat equation with initial data  $n_0$ ,

$$\|n(t)\|_{L^a(\mathbb{R}^d)} \leq C(\|n_0\|_{L^a(\mathbb{R}^d)}) \sqrt{1+t}^{-d/a'}, \quad t \geq 0. \quad (13)$$

Obviously, under the same smallness condition, the same time decay holds true for any  $1 < p \leq a$ ,

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(\|n_0\|_{L^p(\mathbb{R}^d)}) \sqrt{1+t}^{-d/p'}. \quad (14)$$

Also, with the same argument we arrive to (14) for any  $p < \infty$  to the expense of a more restrictive smallness condition  $\|n_0\|_{L^a(\mathbb{R}^d)} + \|\nabla c_0\|_d \leq C(p)$  in (10), that can be dramatic for large  $p$ . Below, we improve this using the regularizing parabolic effect.

#### 4. Hypercontractivity estimate

In order to prove the hypercontractivity estimate (5), using (6) and the properties of the heat kernel  $G$ , we compute

$$\|n(t) - G(t) \star n_0\|_{L^q(\mathbb{R}^d)} \leq C \int_0^t \frac{1}{\sqrt{t-s}^{1+d/\beta}} \|n(s)\|_{L^p(\mathbb{R}^d)} \|\nabla c(s)\|_{L^r(\mathbb{R}^d)} \quad (15)$$

where  $\frac{1}{\beta} = \frac{1}{p} + \frac{1}{r} - \frac{1}{q}$ . Moreover, computing  $\nabla c$  from the representation formula (7), we have

$$\|\nabla c(t)\|_{L^r(\mathbb{R}^d)} \leq C \frac{e^{-t}}{\sqrt{t}^{d(\frac{1}{a}-\frac{1}{r})}} \|\nabla c_0\|_{L^d(\mathbb{R}^d)} + C \int_0^t \frac{e^{(s-t)}}{\sqrt{t-s}^{1+d(\frac{1}{p}-\frac{1}{r})}} \|n(s)\|_{L^p(\mathbb{R}^d)}, \quad t > 0, \quad (16)$$

where  $r \geq d$  and  $r \geq p$ . Let us observe that if  $p = a$ , with  $a$  arbitrarily close to  $\frac{d}{2}$  (in fact  $\frac{d}{2} < a < \frac{2}{3}d$ ), then  $q < d$ . Indeed, the integrability of the functions  $1/\sqrt{t-s}^{(1+d/\beta)}$  and  $1/\sqrt{t-s}^{(1+d(\frac{1}{a}-\frac{1}{r}))}$  in  $s = t$  requires respectively  $\beta > d$  and  $\frac{1}{r} > \frac{1}{a} - \frac{1}{d}$  so that  $\frac{1}{q} > \frac{2}{a} - \frac{2}{d}$ . When  $a = d$ , (5) follows in one step and thus the proof consists in bootstrapping on  $p$ .

*First step:  $p = a$ .* Let us take  $p = a$  and  $d \lesssim \beta < r \lesssim \frac{ad}{d-a}$  so that  $q > a$  and the functions  $1/\sqrt{t-s}^{(1+d(\frac{1}{a}-\frac{1}{r}))}$  and  $1/\sqrt{t-s}^{(1+d/\beta)}$  are integrable in  $s = t$ . Then, since  $\|n(t)\|_{L^a(\mathbb{R}^d)} \leq \|n_0\|_{L^a(\mathbb{R}^d)}$ , the integral term in (16) is bounded and  $\|\nabla c(t)\|_{L^r(\mathbb{R}^d)} = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{r})})$  when  $t \rightarrow 0$ . Inserting this estimate in (15) and using the change of variable  $u = s/t$  we have

$$\|n(t) - G(t) \star n_0\|_{L^q(\mathbb{R}^d)} \leq C \|n_0\|_{L^a(\mathbb{R}^d)} \sqrt{t}^{-d(\frac{1}{\beta}-\frac{1}{r})}, \quad t \rightarrow 0. \quad (17)$$

As a consequence,  $\|n(t)\|_{L^q(\mathbb{R}^d)} = O(\|G(t) \star n_0\|_{L^q(\mathbb{R}^d)}) = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{q})})$  as  $t \rightarrow 0$  because  $n_0 \in L^a(\mathbb{R}^d)$ .

On the other hand, using the estimate (13) in the integral term in (16), we found that

$$\|\nabla c(t)\|_{L^r(\mathbb{R}^d)} \leq C\Gamma \left( \frac{1}{2} - \frac{d}{2a} + \frac{d}{2r} \right) \sqrt{t}^{-d/a'}, \quad t \rightarrow \infty. \quad (18)$$

Inserting this estimate as well as estimate (13) in (15) we obtain for an  $\epsilon > 0$

$$\|n(t) - G(t) \star n_0\|_{L^q(\mathbb{R}^d)} \leq C \sqrt{t}^{-2d/a'+\epsilon}, \quad t \rightarrow \infty. \quad (19)$$

Therefore,  $\|n(t)\|_{L^q(\mathbb{R}^d)} = O(\|G(t) \star n_0\|_{L^q(\mathbb{R}^d)}) = O(1/\sqrt{t}^{d/q'})$  as  $t \rightarrow \infty$ .

*Second step: bootstrap from  $p = a$  to  $q > d$ .* Let us consider an increasing sequence  $p_k$  such that  $p_1 = a$  and a sequence  $r_k$  such that

$$d \leq r_k, \quad p_k \leq r_k, \quad \frac{1}{r_k} > \frac{1}{p_k} - \frac{1}{d}, \quad \frac{1}{p_k} - \frac{1}{p_{k+1}} < \frac{1}{d} - \frac{1}{r_k}.$$

Then the sequence  $\beta_k$  defined by  $\frac{1}{\beta_k} = \frac{1}{p_k} + \frac{1}{r_k} - \frac{1}{p_{k+1}}$  satisfies  $\beta_k > d$ . With this choice we will show in a finite number of steps that  $\|n(t)\|_{L^p(\mathbb{R}^d)}$  behaves as  $\|G(t) \star n_0\|_{L^p(\mathbb{R}^d)}$  in a range of  $p$  going from  $a$  to some  $q > d$ .

(i) Singularity in  $t = 0$ . If  $\|n(t)\|_{L^{p_k}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{p_k})})$  when  $t \rightarrow 0$ , then (16) (with  $p_k$  and  $r_k$  and the change of variable  $u = s/t$ ) gives  $\|\nabla c(t)\|_{L^{r_k}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{r_k})})$  when  $t \rightarrow 0$ . Next, inserting these two singularities in (15) (with  $\beta_k$ ,  $p_k$  and  $r_k$ ) we obtain  $\|n(t) - G(t) \star n_0\|_{L^{p_{k+1}}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{p_{k+1}})})$  as  $t \rightarrow 0$ . (The delicate point here is the singularity at  $s = 0$ ; but we observe that the integrability condition in  $s = 0$  is  $d(\frac{1}{a} - \frac{1}{p_k} + \frac{1}{d} - \frac{1}{r_k}) < 2$  and because  $a > d/2$  this is possible in a range going from  $a$  to some  $p_k > d$ , i.e. for a finite number of steps.) As a consequence  $\|n(t)\|_{L^{p_{k+1}}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d(\frac{1}{a}-\frac{1}{p_{k+1}})})$  as  $t \rightarrow 0$ .

(ii) Decay as  $t \rightarrow \infty$ . Departing from  $\|n(t)\|_{L^{p_k}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d/p'_k})$  as  $t \rightarrow \infty$  in (16), we arrive to  $\|\nabla c(t)\|_{L^{r_k}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d/p'_k})$  as  $t \rightarrow \infty$ . Inserting these two time decays in (15) we find that  $\|n(t) - G(t) \star n_0\|_{L^{p_{k+1}}(\mathbb{R}^d)} = O(1/\sqrt{t}^{2d/p'_k - \epsilon})$  as  $t \rightarrow \infty$ . Finally,  $\|n(t)\|_{L^{p_{k+1}}(\mathbb{R}^d)} = O(1/\sqrt{t}^{d/p'_{k+1}})$  as  $t \rightarrow \infty$ .

*Third step: Hypercontractivity estimate.* Estimate (5) follows taking in (15)  $q = \infty$  and  $p > d$  and working as in the previous steps.

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