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C. R. Acad. Sci. Paris, Ser. I 342 (2006) 635–638
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Combinatorics

Closed walks and eigenvalues of Abelian Cayley graphs

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Received 14 June 2005; accepted 23 February 2006

Available online 29 March 2006

Presented by Jean-Pierre Serre

Abstract

We show that Abelian Cayley graphs contain many closed walks of even length. This implies that given $k \geqslant 3$, for each $\epsilon > 0$, there exists $C = C(\epsilon, k) > 0$ such that for each Abelian group G and each symmetric subset S of G with $1 \notin S$, the number of eigenvalues λ_i of the Cayley graph $X = X(G, S)$ such that $\lambda_i \geqslant k - \epsilon$ is at least $C \cdot |G|$. This can be regarded as an analogue for Abelian Cayley graphs of a theorem of Serre for regular graphs. ***To cite this article: S.M. Cioabă, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Chaînes fermés et valeurs propres des graphes abélien de Cayley. Soit $k \geqslant 3$, pour chaque $\epsilon > 0$, il existe une constante positive $C = C(\epsilon, k) > 0$ telle que pour chaque groupe abélien G et pour chaque sous-ensemble symétrique $S \subset G$ ne contenant pas 1, le nombre de valeurs propres λ_i de graphe de Cayley $X = X(G, S)$ qui satisfont $\lambda_i \geqslant k - \epsilon$ est au moins $C \cdot |G|$. ***Pour citer cet article : S.M. Cioabă, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let X be a k-regular, connected graph on n vertices. Denote by $t_r(u)$ the number of closed walks of length r starting at a vertex u of X and let $\Phi_r(X) = \sum_{u \in X} t_r(u)$. Let $k = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of the adjacency matrix of X. The graph X is called *Ramanujan* if $|\lambda_i| \leqslant 2\sqrt{k-1}$ for each $\lambda_i \neq \pm k$. One of the hardest problems in graph theory is constructing infinite families of k-regular Ramanujan graphs for $k \geqslant 3$ fixed. The only constructions known (see [4,7]) are for $k - 1$ a power of a prime and are obtained from Cayley graphs of certain matrix groups.

J.-P. Serre [4] proved the following result regarding the largest eigenvalues of regular graphs. For a simple proof and related results, see [2,3,8].

Theorem 1.1 *(Serre). For each $\epsilon > 0$ and k, there exists a positive constant $c = c(\epsilon, k)$ such that for any k-regular graph X, the number of eigenvalues λ_i of X with $\lambda_i \geqslant (2 - \epsilon)\sqrt{k-1}$ is at least $c|X|$.*

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.03.005

In this Note, we prove that Abelian Cayley graphs have a large number of closed walks of even length. We use this fact to give a simple proof of the following Serre-type theorem for Abelian Cayley graphs.

Theorem 1.2. *For each $\epsilon > 0$ and k, there exists a positive constant $C = C(\epsilon, k)$ such that for any Abelian group G and for any symmetric set S of elements of G with $|S| = k$ and $1 \notin S$, the number of the eigenvalues λ_i of the Cayley graph $X = X(G, S)$ such that $\lambda_i \geqslant k - \epsilon$ is at least $C \cdot |G|$.*

Cayley graphs are defined as follows. Let G be a finite multiplicative group, with identity 1 and suppose S is a subset of G such that $1 \notin S$ and $s \in S$ implies $s^{-1} \in S$. The Cayley graph $X = X(G, S)$ is the graph with vertex set G and with $x, y \in G$ adjacent if $xy^{-1} \in S$. Notice that adjacency is well-defined since S is symmetric. Also, G is regular with valency $k = |S|$ and it contains no loops since $1 \notin S$. It is easy to see that X is connected if and only if S generates G. If G is an Abelian group and S is a symmetric subset of k elements of G, then the eigenvalues of $X(G, S)$ are $\lambda_\chi = \sum_{s \in S} \chi(s)$ where χ ranges over all the characters of G (see Li [9]). This fact was used by Friedman, Murty and Tillich [6] who proved that the second largest eigenvalue of a k-regular Abelian Cayley graph with n vertices is at least $k - \mathrm{O}(kn^{-4/k})$.

There are Abelian Cayley graphs that are Ramanujan (see Li [9]). The proof that these graphs are Ramanujan is often based on number theoretic estimates of character sums. For each choice of a degree of regularity, all these constructions produce only a *finite* number of Ramanujan graphs. Theorem 1.2 shows that it is impossible to construct an infinite family of constant degree Abelian Cayley graphs that are Ramanujan. This also follows from [1] and [6].

2. Closed walks in Abelian Cayley graphs

Let G be a finite Abelian group. There is a simple bijective correspondence between the closed walks of length r starting at a vertex u of a Cayley graph $X(G, S)$ and the r-tuples $(a_1, \ldots, a_r) \in S^r$ with $\prod_{i=1}^r a_i = 1$. To each closed walk $u = u_0, u_1, \ldots, u_{r-1}, u_r = u$, we associate the r-tuple

$$\left(u_0u_1^{-1}, u_1u_2^{-1}, \ldots, u_{r-2}u_{r-1}^{-1}, u_{r-1}u_r^{-1}\right) \in S^r.$$

Suppose that

$$S = \left\{x_1, x_2, \ldots, x_s, x_1^{-1}, x_2^{-1}, \ldots, x_s^{-1}, y_1, \ldots, y_t\right\}, \tag{1}$$

where each x_i has order greater than 2 for $1 \leqslant i \leqslant s$ and each y_j has order 2 for $1 \leqslant j \leqslant t$. The degree of the Cayley graph of G with respect to S is $k = 2s + t$. Let $W_{2r}(X)$ be the number of $2r$-tuples from S^{2r} in which the number of appearances of x_i is the same as the number of appearances of x_i^{-1} for all $1 \leqslant i \leqslant s$ and the number of appearances of y_j is even for all $1 \leqslant j \leqslant t$. More precisely, $W_{2r}(X)$ counts $2r$-tuples from S^{2r} in which p positions are occupied by x_i's, p positions are occupied by x_i^{-1}'s and the remaining $2r - 2p$ positions are occupied by y_j's (each of them appearing an even number of times), where $0 \leqslant p \leqslant r$. These choices imply that the product of the $2r$ elements in this type of $2r$-tuple is 1.

Thus, $t_{2r}(u) \geqslant W_{2r}(X)$ for each $u \in V(X)$. This implies

$$\Phi_{2r}(X) = \sum_{u \in V(X)} t_{2r}(u) \geqslant nW_{2r}(X), \tag{2}$$

for each $r \geqslant 1$.

We evaluate $W_{2r}(X)$ by choosing first the $2p$ positions for the x_i's and their inverses. This can be done in $\binom{2r}{2p}$ ways. Then we choose p positions for the x_i's. This is done in $\binom{2p}{p}$ ways and the rest are left for x_i^{-1}'s. Since this happens for all $0 \leqslant p \leqslant r$, we get the following expression for $W_{2r}(X)$

Lemma 2.1. *For each $r \geqslant 1$, we have*

$$W_{2r}(X) = \sum_{p=0}^{r} \binom{2r}{2p}\binom{2p}{p} \sum_{i_1+\cdots+i_s=p} \binom{p}{i_1, \ldots, i_s}^2 \sum_{2j_1+\cdots+2j_t=2r-2p} \binom{2r-2p}{2j_1, \ldots, 2j_t}.$$

We now obtain lower bounds for

$$c(p,s)=\sum_{i_1+\cdots+i_s=p}\binom{p}{i_1,\ldots,i_s}^2 \quad\text{and}\quad d(r-p,t)=\sum_{2j_1+\cdots+2j_t=2r-2p}\binom{2r-2p}{2j_1,\ldots,2j_t}.$$

Obtaining a closed formula for any of these two sums seems to be an interesting and difficult combinatorial problem in itself. We use the Cauchy–Schwarz inequality to obtain a lower bound on $c(m,l)$.

$$c(m,l)=\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}^2 \geqslant \frac{\left(\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}\right)^2}{\binom{m+l-1}{l-1}}=\frac{l^{2m}}{\binom{m+l-1}{l-1}}.$$

Our lower bound for $d(m,l)$ follows from the following result of Fixman [5].

$$d(m,l)=2^{-l}\sum_{j=0}^{l}\binom{l}{j}(l-2j)^{2m}>2^{1-l}l^{2m}.$$

Hence, we have

$$c(m,l)>\frac{l^{2m}}{\binom{m+l-1}{l-1}},\qquad d(m,l)>\frac{l^{2m}}{2^{l-1}}. \tag{3}$$

These two inequalities and Lemma 2.1 easily imply the next result. Recall that $k=2s+t$.

Lemma 2.2. *For each $r\geqslant 1$, we have*

$$W_{2r}(X)>\frac{k^{2r}}{2^k(2r+1)\binom{k+r-1}{k-1}}.$$

Proof. Using Lemma 2.1, inequalities (3) and $\binom{2p}{p}>\frac{2^{2p}}{2p+1}$, we get

$$\begin{aligned}
W_{2r}(X)&=\sum_{p=0}^{r}\binom{2r}{2p}\binom{2p}{p}c(p,s)d(r-p,t)>\sum_{p=0}^{r}\binom{2r}{2p}\frac{2^{2p}}{2p+1}\cdot\frac{s^{2p}}{\binom{s+p-1}{s-1}}\cdot\frac{t^{2r-2p}}{2^{t-1}}\\
&>\sum_{p=0}^{r}\binom{2r}{2p}\frac{(2s)^{2p}t^{2r-2p}}{(2r+1)\binom{s+r-1}{s-1}2^{k-1}}>\frac{1}{(2r+1)\binom{k+r-1}{k-1}2^{k-1}}\sum_{p=0}^{r}\binom{2r}{2p}(2s)^{2p}t^{2r-2p}\\
&=\frac{1}{(2r+1)\binom{k+r-1}{k-1}2^{k-1}}\cdot\frac{(2s+t)^{2r}+(2s-t)^{2r}}{2}>\frac{k^{2r}}{2^k(2r+1)\binom{k+r-1}{k-1}}. \qquad \square
\end{aligned}$$

3. The proof of Theorem 1.2

We now present the proof of Theorem 1.2.

Proof. Let $\epsilon>0$. Consider an Abelian group G and S a subset of G of size k. Denote by n the order of G and by m the number of eigenvalues λ_i of $X=X(G,S)$ such that $\lambda_i\geqslant k-\epsilon$. Then there are exactly $n-m$ eigenvalues of X that are less than $k-\epsilon$. It follows that

$$\operatorname{tr}(kI+A)^{2l}=\sum_{i=1}^{n}(k+\lambda_i)^{2l}<(n-m)(2k-\epsilon)^{2l}+m(2k)^{2l}, \tag{4}$$

for each $l\geqslant 1$.

Using Lemma 2.2 and (2), we obtain the following

$$\begin{aligned}\operatorname{tr}(kI+A)^{2l} &= \sum_{i=0}^{2l}\binom{2l}{i}k^i\Phi_{2l-i}(X) \geqslant \sum_{j=0}^{l}\binom{2l}{2j}k^{2j}\Phi_{2l-2j}(X)\\ &> n\sum_{j=0}^{l}\binom{2l}{2j}k^{2j}\frac{k^{2l-2j}}{2^k(2(l-j)+1)\binom{k+l-j-1}{k-1}} > n\frac{(2k)^{2l}}{2^{k+1}(2l+1)\binom{k+l-1}{k-1}}\end{aligned}$$

for each $l \geqslant 1$. Combining this inequality with (4), it follows that

$$\frac{m}{n} > \frac{\frac{1}{2^{k+1}(2l+1)\binom{k+l-1}{k-1}}(2k)^{2l}-(2k-\epsilon)^{2l}}{(2k)^{2l}-(2k-\epsilon)^{2l}}, \tag{5}$$

for each $l \geqslant 1$. Now

$$\lim_{l\to\infty}\sqrt[2l]{\frac{1}{2^{k+1}(2l+1)\binom{k+l-1}{k-1}}(2k)^{2l}} = 2k > 2k-\epsilon = \lim_{l\to\infty}\sqrt[2l]{2(2k-\epsilon)^{2l}}.$$

This implies that there exists $l_0 = l(\epsilon, k)$ such that

$$\frac{1}{2^{k+1}(2l+1)\binom{k+l-1}{k-1}}(2k)^{2l}-(2k-\epsilon)^{2l} > (2k-\epsilon)^{2l},$$

for each $l \geqslant l_0$. Letting

$$C(\epsilon,k) = \frac{(2k-\epsilon)^{2l_0}}{(2k)^{2l_0}-(2k-\epsilon)^{2l_0}} > 0$$

it follows that

$$\frac{m}{n} > C(\epsilon,k).$$

This proves the theorem. □

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