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Topology

Cohomology of singular Riemannian foliations

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Abstract

Associated to a smooth foliation $(M,\mathcal{F})$ are defined the *basic* and the *foliated* cohomologies. These cohomologies are related to the de Rham cohomology by the *de Rham spectral sequence* of $\mathcal{F}$, $E_{r,dR}^{s,t}(M)$, constructed by filtering the de Rham complex of the manifold. For a Riemannian foliation on a compact manifold the second term of this spectral sequence, $E_{2,dR}^{s,t}(M)$, is finite dimensional and a topological invariant.

In this Note we prove these two results, fitness dimension and topological invariance, for the cohomology of *singular* Riemannian foliations. The proof uses the previous theorems for the regular case and the structure of singular Riemannian foliations described by P. Molino. For the basic cohomology these results have been proved by R. Wolak. ***To cite this article: X.M. Masa, A. Rodríguez-Fernández, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Cohomologie des feuilletages riemanniens singuliers. La filtration du complexe de De Rham d'une variété feuilletée $(M,\mathcal{F})$ définit une suite spectrale $E_{r,dR}^{s,t}(M)$ reliant les cohomologies *basique* et *feuilletée* du feuilletage $\mathcal{F}$ à la cohomologie de De Rham de la variété ambiance. Pour un feuilletage riemannien d'une variété compacte le deuxième terme de cette suite spectrale est un invariant topologique de dimension finie.

Dans cette Note, nous prouvons la finitude et l'invariance topologique de ce terme pour les feuilletages riemanniens *singuliers*. La preuve combine les résultats du cas régulier avec la description de la structure des feuilletages riemanniens singuliers faite par P. Molino. Pour la cohomologie basique, ces résultats ont été prouvés par R. Wolak. ***Pour citer cet article : X.M. Masa, A. Rodríguez-Fernández, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

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1. Preliminaries

1.1. Singular Riemannian foliations (see [3, Chapter 6])

We assume that the manifold M is compact. A singular Riemannian foliation on M is a partition $\mathcal{F}$ of M by connected immersed submanifolds, the leaves, such that the module of smooth fields tangent to the leaves is transitive on each leaf and there exists a Riemannian metric that is adapted to $\mathcal{F}$ in the sense that every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

We denote by $\overline{\mathcal{F}}$ the partition of M by leaf closures and by $M/\overline{\mathcal{F}}$ the corresponding quotient space. The map $p:M\to M/\overline{\mathcal{F}}$ has the following good local property. Let x be a point of $M/\overline{\mathcal{F}}$, $X=p^{-1}(x)$. The foliation $\mathcal{F}$ defines on X a regular Riemannian foliation $\mathcal{F}_X$ with dense leaves and X has a base of saturated tubular neighborhoods, $\widetilde{X}\times_\pi V$, quotient of $\widetilde{X}\times V$ by the diagonal action of $\pi=\pi_1(X)$, where $\widetilde{X}$ is the universal covering space of X and V is an open set of $\mathbb{R}^k$, k the codimension of X in M, and where π acts on V by linear isometries.

The homothetic lemma [3] states that X and $\widetilde{X}\times_\pi V$ are homotopically equivalent by homotopies that preserve the foliation, in the sense that each leaf goes into a leaf for each parameter of the homotopy. The homotopy equivalences are the protection $p:\widetilde{X}\times_\pi V\to X$ and the zero section $s_0:X\to\widetilde{X}\times_\pi V$.

The open sets $p(\widetilde{X}\times_\pi V)$ will be called *adapted open sets* in $M/\overline{\mathcal{F}}$. A cover by adapted open sets is said to be *good* if any finite nonempty intersection is also adapted (for its existence, see [4, Proposition 1]).

1.2. The Čech complex

Let W be a topological space, $\mathcal{L}$ a sheaf on W and $\mathfrak{U}=\{U_\alpha;\ \alpha\in I\}$ an open cover of W. We shall use σ^n to denote an ordered n-simplex $\langle\alpha_0,\dots,\alpha_n\rangle$ of the nerve $N(\mathfrak{U})$ of $\mathfrak{U}$, and U_{σ^n} to denote $U_{\alpha_0,\dots,\alpha_n}=U_{\alpha_o}\cap\dots\cap U_{\alpha_n}$. We define the Čech complex of $\mathfrak{U}$ with coefficients in $\mathcal{L}$ as

$$\check{C}^n(\mathfrak{U};\mathcal{L})=\prod_{\sigma^n}\mathcal{L}(U_{\sigma^n}),$$

with the usual differential

$$\delta:C^n(\mathfrak{U},\mathcal{L})\longrightarrow C^{n+1}(\mathfrak{U},\mathcal{L}),$$
$$(\delta w)_{\alpha_0\cdots\alpha_{n+1}}=\sum_{i=0}^{n+1}(-1)^i j_i^* w_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{n+1}},$$

where j_i denotes the inclusion $U_{\langle\alpha_0,\dots,\alpha_{n+1}\rangle}\to U_{\langle\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_{n+1}\rangle}$.

If we start with a differential sheaf $(\mathcal{L}^*,d)$, we get a double complex $\check{C}^*(\mathfrak{U};\mathcal{L}^*)$ and two associated spectral sequences,

$${}_IE_2^{s,t}=\check{H}^s\big(\mathfrak{U},\mathcal{H}^t\big(\mathcal{L}^*\big)\big)\quad\text{and}\quad {}_{II}E_2^{s,t}=H^s\big(H^t\big(\mathfrak{U},\mathcal{L}^*\big)\big).$$

We are interested in the case where $\mathcal{L}^*$ is acyclic, then $H^t(\mathfrak{U},\mathcal{L}^*)=H^t(W,\mathcal{L}^*)=0$ for $t>0$ and we have

$${}_IE_2^{s,t}=\check{H}^s\big(\mathfrak{U},\mathcal{H}^t\big(\mathcal{L}^*\big)\big)\Rightarrow H^{s+t}\big(\Gamma\big(\mathcal{L}^*\big)\big). \tag{1}$$

1.3. Alexander–Spanier spectral sequence [2]

Let X be a topological space, and X' the same set as X with a finer topology. Let U be an open set in X. A map

$$\varphi:U^{p+1}\to\mathbb{R}$$

is said to be a *basic Alexander–Spanier p-cochain* in U if it is locally constant when one considers in U^{p+1} the topology induced by X'.

For each U, the vector space of basic Alexander–Spanier cochains in U, with the obvious restriction maps, defines a presheaf, which generates the *sheaf of basic Alexander–Spanier cochains* $AS^*_{(X'|X)}$. With the usual differential

$$\delta\varphi(x_0,\dots,x_p)=\sum_{i=0}^{p}(-1)^i\varphi(x_0,\dots,\hat{x}_i,\dots,x_p) \tag{2}$$

we have a resolution of the constant sheaf $\mathbb{R}_X$ and the *Alexander–Spanier spectral sequence* of $(X'|X)$,

$$E_2^{p,q}(X' \mid X) = H^p H^q\big(X, AS^{\star}_{(X'|X)}\big) \Rightarrow H^{p+q}(X, \mathbb{R}). \tag{3}$$

One can consider only continuous or differentiable Alexander–Spanier cochains, to get a continuous or differentiable resolution, respectively, and the corresponding spectral sequences. Let $(M, \mathcal{F})$ be a foliated manifold, $M = \bigcup_{x\in M} L_x$. Denote by $M^{\mathcal{F}}$ the set M with the leaf topology: a basis is formed by the connected components of intersections of open sets of M with leaves. The Alexander–Spanier sheaf and the spectral sequence of the foliated manifold will be the one associated to $(M^{\mathcal{F}} \mid M)$. We use the notation

$${}_{\mathrm{cont}}AS^*_{\mathcal{F}}, \qquad {}_{\mathrm{diff}}AS^*_{\mathcal{F}}$$

for the continuous and differentiable Alexander–Spanier sheaves, respectively, and

$$E^{p,q}_{r,\mathrm{cont}}(M) \quad \text{and} \quad E^{p,q}_{r,\mathrm{diff}}(M)$$

for the corresponding spectral sequences.

2. Finiteness of $E_{2,dR}(M)$

The de Rham spectral sequence of $\mathcal{F}$, $E^{p,q}_{r,dR}(M)$, can be obtained from the resolution of the constant sheaf $\mathbb{R}_M$ given by the sheaves of basic forms, $(\mathcal{A}_{\mathcal{F}}, d)$. In fact, $E^{p,q}_{r,dR}(M) \cong H^p H^q(M, \mathcal{A}^*_{\mathcal{F}})$.

Let $\mathcal{H}^q(p, \mathcal{A}^t_{\mathcal{F}})$ be the sheaf in $M/\overline{\mathcal{F}}$ defined by

$$\mathcal{H}^q\big(p, \mathcal{A}^t_{\mathcal{F}}\big)(U) = H^q\big(p^{-1}(U), \mathcal{A}^t_{\mathcal{F}}\big)$$

for an open set $U \subset M/\overline{\mathcal{F}}$. It is the derived sheaf of $\mathcal{A}^t_{\mathcal{F}}$. With the differential induced by d, $\mathcal{H}^q(p, \mathcal{A}^*_{\mathcal{F}})$ is a differential sheaf. As $\mathcal{H}^0(p, \mathcal{A}^0_{\mathcal{F}})$ is soft, and the sheaves $\mathcal{H}^q(p, \mathcal{A}^t_{\mathcal{F}})$ are $\mathcal{H}^0(p, \mathcal{A}^0_{\mathcal{F}})$-modules, they are acyclic.

Theorem 2.1. *The second term of the de Rham spectral sequence of a singular Riemannian foliation is finite dimensional.*

We consider a finite good cover $\mathfrak{U}$ of $M/\overline{\mathcal{F}}$. We consider the double complex $\check{C}^s(\mathfrak{U}, \mathcal{H}^q(p, \mathcal{A}^t_{\mathcal{F}}))$ and the spectral sequence

$$E_2^{s,t} = \check{H}^s\big(\mathfrak{U}, \mathcal{H}^t\big(\mathcal{H}^q\big(p, \mathcal{A}^*_{\mathcal{F}}\big)\big)\big) \Rightarrow E^{s+t,q}_{2,dR}(M).$$

Now, for any U_σ, $\sigma \in N(\mathfrak{U})$, denote by X_σ the leaf closure such that $p^{-1}(U_\sigma)$ is homotopically equivalent to it. We have

$$\mathcal{H}^t\big(\mathcal{H}^q\big(p, \mathcal{A}^*_{\mathcal{F}}\big)\big)(U) = E^{t,q}_{2,dR}\big(p^{-1}(U)\big) \cong E^{t,q}_{2,dR}(X_\sigma).$$

The foliation on X_σ being regular, $E^{t,q}_{2,dR}(X)$ is finite dimensional [1]. As a consequence, already the complex $\check{C}^s(\mathfrak{U}, \mathcal{H}^t(\mathcal{H}^q(p, \mathcal{A}^*_{\mathcal{F}})))$ is finite dimensional, so $\check{H}^s(\mathfrak{U}, \mathcal{H}^t(\mathcal{H}^q(p, \mathcal{A}^*_{\mathcal{F}})))$ and $E_2^{s+t,q}(\mathcal{F})$ are finite dimensional.

3. Topological invariance of $E_2(\mathcal{F})$

Theorem 3.1. *The de Rham spectral sequence of a singular Riemannian foliation is a topological invariant from* E_2 *onwards.*

For each differential sheaf $\mathcal{A}^*_{\mathcal{F}}$, ${}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}}$ and ${}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}}$, and each $q \geqslant 0$, we have three spectral sequences of the type (1), constructed with the Čech complex associated to a good cover $\mathfrak{U}$ of $M/\overline{\mathcal{F}}$ and to the derived sheaves $\mathcal{H}^q(p, \mathcal{A}^*_{\mathcal{F}})$, $\mathcal{H}^q(p, {}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}})$ and $\mathcal{H}^q(p, {}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}})$. These spectral sequences are

$$\check{H}^s\big(\mathfrak{U}, \mathcal{H}^t\big(\mathcal{H}^q\big(p, \mathcal{A}^*_{\mathcal{F}}\big)\big)\big) \Rightarrow E^{s+t,q}_{2,dR}(M),$$
$$\check{H}^s\big(\mathfrak{U}, \mathcal{H}^t\big(\mathcal{H}^q\big(p, {}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}}\big)\big)\big) \Rightarrow E^{s+t,q}_{2,\mathrm{diff}}(M),$$

and

$$\check{H}^s\big(\mathfrak{U}, \mathcal{H}^t\big(\mathcal{H}^q\big(p, {}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}}\big)\big)\big) \Rightarrow E_{2,\mathrm{cont}}^{s+t,q}(M).$$

We are going to prove that all of them are isomorphic to each other from E_1 onwards. But the latter, constructed with ${}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}}$, is obviously invariant by homeomorphisms, and so the theorem follows.

Let $I : {}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}} \to {}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}}$ be the inclusion map. We define another homomorphism of differential sheaves $\Lambda : {}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}} \to \mathcal{A}^*_{\mathcal{F}}$ in a standard way: for an open set U of M, for $x \in U$ and $Z_1, \ldots, Z_p \in T_x M$,

$$\Lambda(\varphi)_x(Z_1, \ldots, Z_p) = \frac{1}{p!} \sum_{\tau \in \mathcal{S}_p} \mathrm{sgn}(\tau) \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_p} \varphi(x, \exp_x \varepsilon_1 Z_{\tau(1)}, \ldots, \exp_x \varepsilon_p Z_{\tau(p)})|_{\varepsilon_i = 0},$$

where $\varepsilon_i \in \mathbb{R}$, $1 \leqslant i \leqslant p$.

The homomorphisms I and Λ induced isomorphisms between the sheaf $\mathcal{H}^t(\mathcal{H}^q(p, {}_{\mathrm{diff}}\mathcal{AS}^*_{\mathcal{F}})$ and the sheaves $\mathcal{H}^t(\mathcal{H}^q(p, \mathcal{A}^*_{\mathcal{F}})$ and $\mathcal{H}^t(\mathcal{H}^q(p, {}_{\mathrm{cont}}\mathcal{AS}^*_{\mathcal{F}})$, respectively. To prove that, we compute their stacks. Let U be an adapted open set such that $p^{-1}(U)$ is a tubular neighborhood of $X = p^{-1}(x)$. By the same argument of the proof of Theorem 1, we see that the space of sections over U of these sheaves are

$$E_{2,\mathrm{diff}}^{t,q}\big(p^{-1}(U)\big) \cong E_{2,\mathrm{diff}}^{t,q}(X), \qquad E_{2,dR}^{t,q}\big(p^{-1}(U)\big) \cong E_{2,dR}^{t,q}(X),$$

and

$$E_{2,\mathrm{cont}}^{t,q}\big(p^{-1}(U)\big) \cong E_{2,\mathrm{cont}}^{t,q}(X).$$

But on X the foliation is Riemannian regular, so the homomorphisms I and Λ induce isomorphisms between the second terms of the spectral sequences [2].

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