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C. R. Acad. Sci. Paris, Ser. I 342 (2006) 411–416
COMPTES RENDUS MATHEMATIQUE
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Algebraic Geometry/Differential Geometry

Kähler manifolds with numerically effective Ricci class and maximal first Betti number are tori

Fuquan Fang [1]

Department of Mathematics, Capital Normal University, Beijing 100037, PR China
Chern Institute of Mathematics, Nankai University, Tianjin 300071, PR China

Received 11 May 2005; accepted after revision 16 November 2005

Available online 27 January 2006

Presented by Jean-Pierre Demailly

Abstract

Let M be a n-dimensional Kähler manifold with numerically effective Ricci class $c_1(M)$. In this Note we prove that, if the first Betti number $b_1(M)$ is equal to $2n$, then M is biholomorphic to a n-dimensional complex torus. ***To cite this article: F. Fang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Les variétés kählériennes de classe de Ricci numériquement effective et de premier nombre de Betti maximal sont les tores. Soit M une variété kählérienne compacte de dimension n et de classe de Ricci $c_1(M)$ numériquement effective. Dans cette note nous montrons que si le premier nombre de Betti $b_1(M)$ est égal à $2n$, alors M est biholomorphe à un tore complexe de dimension n. ***Pour citer cet article : F. Fang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit M une variété complexe compacte munie d'une métrique hermitienne ω. D'après [5,6], un fibré holomorphe L sur M est dit *numériquement effectif* (nef) si pour tout $\varepsilon > 0$, il existe une métrique hermitienne lisse h_ε sur L telle que sa courbure satisfait

$$\Theta_{h_\varepsilon} \geqslant -\varepsilon\omega.$$

Si M est projective, L est nef quand $L \cdot C \geqslant 0$ pour toutes courbes $C \subset M$. On dit qu'une variété kählerienne M est nef si le fibré anticanonique $-K_M$ est nef. Dans [5], il est conjecturé que, pour une variété kählerienne nef M, on a :

(A1) le groupe fondamental $\pi_1(M)$ est de croissance polynomiale,
(A2) l'application d'Albanese $\alpha : M \to Alb(M)$ est surjective.

E-mail address: ffang@nankai.edu.cn (F. Fang).
[1] Supported partially by Max-Planck Institute für Mathematik, IHES and a 973 Project for Foundation Science of China.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.11.019

Si M est projective, (A2) a été démontrée par Zhang [15]. Dans [11] Paun a montré (A1) en admettant une conjecture de Gromov concernant le groupe fondamental des variétés riemanniennes de courbure de Ricci presque non-negative (voir aussi [3]).

En utilisant le théorème d'Aubin–Calabi–Yau [1,14], il est montré dans [5] que M est nef si et seulement si il existe une suite de métriques kählerienne $\{\omega_k\}$ sur M telles que, pour tout $k > 0$, la métrique ω_k appartienne à une classe de cohomology fixée $\{\omega\}$, et la courbure de Ricci de ω_k est minorée par $-1/k$.

Un théorème de type de Bochner pour le premier nombre de Betti a été obtenu par Paun [12]. Il affirme que pour toute variété kählerienne nef M de dimension complexe n, on a $b_1(M) \leqslant 2n$.

Le résultat principal de cette Note est :

Théorème 0.1. *Soit M une variété kählerienne nef de dimension complexe n. Si le premier nombre de Betti $b_1(M) = 2n$, alors M est biholomorphe à un tore complexe de dimension n.*

On peut considérer le Théorème 1.1 comme une version complexe d'une conjecture de Gromov, démontrée par Colding [4], qui dit qu'une variété riemannienne de dimension n de courbure de Ricci presque non-negative et dont le premier nombre de Betti est égal à n, est homeomorphe au tore T^n.

Évidement le Théorème 1.1 implique la conjecture (A2) dans le cas où $b_1(M) = 2n$. En fait, la preuve du Théorème 1.1 fournit une borne supérieure uniforme pour les diametres. Mais cette estimation n'est pas vraie si $b_1(M) = 2n - 2$ car le premier nombre de Betti d'une variété kählerienne est toujours pair. Comme dans la même classe de Kähler, il existe une suite de métriques kähleriennes sur S^2, de courbure de Ricci positive, telle que les espaces associés convergent vers un espace topologique non-compact de dimension 1, la variété produit $T_{\mathbb{C}}^{n-1} \times S^2$ nous donne un contre-exemple.

Le résultat ci-dessous vérifie la condition (A2) pour les variétés de $b_1(M) = 2n - 2$, tandis que le rang de $G_1/[G_1, G]$ est au moins deux, où $G = \pi_1(M)$ et $G_1 = [G, G]$.

Théorème 0.2. *Soit M une variété kählerienne nef de dimension n. Posons $G = \pi_1(M)$. Si $b_1(M) = 2n - 2$, et si le rang de $G_1/[G_1, G]$ est au moins deux, où $G_1 = [G, G]$, alors l'application d'Albanese $\alpha : M \to T_{\mathbb{C}}^{n-1}$ est surjective.*

Remarque 1. Le Théorème 1.1 entraine la conjecture (A2) pour $n = 2$. Ceci a été obetenu précédentement dans [5] par des méthodes de géométrie algébrique.

Les démonstrations de nos Théorèmes utilisent des résultats très profonds de géométrie Riemannienne, y compris la convergence équivariante de Gromov–Hausdorff [7], le théorème de scindage de Cheeger–Colding pour les espaces limites [3], et un résultat de Paun [12]. Il serait intéressant de démontrer le Théorème 1.1 par des méthodes de géométrie algébrique pure. En fait, si l'application d'Albanese α est surjective, en utilisant l'équation de Poincaré–Lelong, on peut obtenir facilement qu'une variété kählerienne nef M de dimension n et telle que $b_1(M) = 2n$ est biholomorphe à un tore complexe $T_{\mathbb{C}}^n$ (voir aussi [10]).

D'après Campana [2], les conjectures (A1) and (A2) ci-dessus et le théorème célèbre de Gromov [8] impliquent que le groupe fondamental d'une variété kählerienne nef est presque abelien. Notre approche suggère la conjecture ci-dessous :

Conjecture 0.3. *Soit M une variété kählerienne nef de dimension n. Si il existe un epimorphisme $\varphi : \pi_1(M) \to \Gamma$, où Γ est un groupe nilpotent sans torsion de rang au moins $2n$, alors $\Gamma \cong \mathbb{Z}^{2n}$ et M est biholomorphe à un tore complexe de dimension n.*

1. Introduction

Let M be a compact complex manifold with a fixed hermitian metric ω. By [5,6] a holomorphic line bundle L over M is called *numerically effective* (abb. nef) if, for every $\varepsilon > 0$, there is a smooth Hermitian metric h_ε on L such that the curvature satisfies:

$$\Theta_{h_\varepsilon} \geqslant -\varepsilon\omega.$$

If M is projective, L is nef precisely if $L \cdot C \geqslant 0$ for all curves $C \subset M$. We say a Kähler manifold M is nef if the anticanonical bundle $-K_M$ is nef. In [5] it is conjectured, for a nef Kähler manifold M, both of the following hold:

(A1) the fundamental group $\pi_1(M)$ has polynomial growth;
(A2) the Albanese map $\alpha : M \to Alb(M)$ is surjective.

If M is projective, (A2) was proved by Zhang [15]. In [11] Paun proved (A1), assuming a conjecture of Gromov concerning the fundamental group of Riemannian manifold with almost non-negative Ricci curvature (compare [3]).

By the Aubin–Calabi–Yau theorem [1,14], [5] proved that M is nef if and only if there exist a sequence of Kähler metrics $\{\omega_k\}$ on M such that, for each $k > 0$, the metric ω_k belongs to a fixed cohomology class $\{\omega\}$, and the Ricci curvature of ω_k is bounded from below by $-1/k$.

A Bochner type theorem for the first Betti number was obtained by Paun [12], namely, for every nef Kähler manifold M of complex dimension n it holds that $b_1(M) \leqslant 2n$. The main result of this Note is the following:

Theorem 1.1. *Let M be a nef Kähler manifold of dimension n. If the first Betti number $b_1(M) = 2n$, then M is biholomorphic to a complex torus of dimension n.*

Theorem 1.1 may be considered as a complex version of a conjecture of Gromov, proved by Colding [4], which asserts that a Riemannian n-manifold of almost non-negative Ricci curvature and first Betti number n is homeomorphic to the torus T^n.

Obviously, Theorem 1.1 implies conjecture (A2) in the case of $b_1(M) = 2n$. In the proof of Theorem 1.1, in fact we will prove that there is a uniform upper bound for the diameters. But this does not hold if $b_1(M) = 2n - 2$ (the first Betti number of a Kähler manifold is always even). Since there is a sequence of Kähler metrics on S^2 in the same Kähler class with positive Ricci curvature but converge to a non-compact space of dimension 1, thus the product $T_{\mathbb{C}}^{n-1} \times S^2$ serves as an example.

The following result verifies (A2) for manifold with $b_1(M) = 2n - 2$, provided $G_1/[G_1, G]$ has rank at least two, where $G = \pi_1(M)$, $G_1 = [G, G]$.

Theorem 1.2. *Let M be a nef Kähler manifold of dimension n. Let $G = \pi_1(M)$. If the first Betti number $b_1(M) = 2n - 2$, and $G_1/[G_1, G]$ has rank at least two where $G_1 = [G, G]$, then the Albanese map $\alpha : M \to T_{\mathbb{C}}^{n-1}$ is surjective.*

Remark 1. By Theorem 1.1 one confirms immediately conjecture (A2) for $n = 2$. This was first obtained in [5] using algebraic geometry methods.

The proof of our theorems uses the deep results in Riemannian geometry, including the equivariant Gromov–Hausdorff convergence [7], a splitting theorem of Cheeger–Colding for limit spaces [3], and a result of Paun [12]. It would be interesting if Theorem 1.1 could be proved using pure algebraic geometry. Indeed, if the Albanese map α is surjective, by the Poincaré–Lelong equation, one obtains easily that a nef Kähler manifold M of dimension n with $b_1(M) = 2n$ is biholomorphic to the complex torus $T_{\mathbb{C}}^n$ (compare [10]).

By Campana [2] the above conjectures (A1) and (A2) together with Gromov's celebrated theorem [8] imply that the fundamental group of a nef Kähler manifold is almost Abelian. By our approach, it seems plausible to prove the following:

Conjecture 1.3. *Let M be a nef Kähler manifold of dimension n. If there is an epimorphism $\varphi : \pi_1(M) \to \Gamma$ where Γ is a torsion free nilpotent group of rank at least $2n$, then $\Gamma \cong \mathbb{Z}^{2n}$ and M is biholomorphic to a complex torus of dimension n.*

2. Proof of Theorems 1.1 and 1.2

By [5], a nef Kähler manifold M admits a family of Kälher metrics ω_ε in the same Kähler class $[\omega]$ with Ricci curvature $\mathrm{Ric}(\omega_\varepsilon) \geqslant -\varepsilon\omega$, where $\varepsilon \in (0, 1)$. The diameters of this family may not have a uniform upper bound. In other words, the pointed Gromov–Hausdorff limit of (M, ω_ε) may not be compact. Because of this, many techniques in metric geometry do not apply to this situation. To overcome this difficulty, the following key lemma was obtained, [5].

Lemma 2.1 ([5]). *Let M be a nef Kähler manifold. Let $U \subset \widetilde{M}$ (the universal covering of M) be a compact subset. Then, $\forall \delta > 0$, there exists a closed subset $U_{\varepsilon,\delta} \subset U$ such that*

$$\mathrm{vol}_{\omega}(U - U_{\varepsilon,\delta}) < \delta; \qquad \mathrm{diam}_{\omega_\varepsilon}(U_{\varepsilon,\delta}) \leqslant C/\sqrt{\delta}, \tag{1}$$

where C is a constant independent of ε and δ.

For convenience let us recall the definition of equivariant Gromov–Hausdorff distance (cf. [7] for details).

Let $\mathfrak{M}$ (resp. $\mathfrak{M}_{\mathrm{eq}}$) denote the set of all isometry classes of pointed metric spaces (X, p) (resp. triples (X, Γ, p)), such that, for any D, the metric ball $B_p(D, X)$ of radius D is relatively compact and such that X is a length space [9,7] (resp. $(X, p) \in \mathfrak{M}$ and Γ is a closed subgroup of isometries of X).

Let $\Gamma(D) = \{\gamma \in \Gamma\colon\ d(\gamma p, p) < D\}$.

Definition 2.2. Let $(X, \Gamma, p), (Y, \Lambda, q) \in \mathfrak{M}_{\mathrm{eq}}$. An ε-equivariant pointed Hausdorff approximation stands for a triple (f, φ, ψ) of maps $f : B_p(\frac{1}{\varepsilon}, X) \to Y$, $\varphi : \Gamma(\frac{1}{\varepsilon}) \to \Lambda(\frac{1}{\varepsilon})$, and $\psi : \Lambda(\frac{1}{\varepsilon}) \to \Gamma(\frac{1}{\varepsilon})$ such that

(S1) $f(p) = q$;

(S2) the ε-neighborhood of $f(B_p(\frac{1}{\varepsilon}, X))$ contains $B_q(\frac{1}{\varepsilon}, Y)$;

(S3) if $x, y \in B_p(\frac{1}{\varepsilon}, X)$, then $|d(f(x), f(y)) - d(x, y)| < \varepsilon$;

(S4) if $\gamma \in \Gamma(\frac{1}{\varepsilon})$, $x \in B_p(\frac{1}{\varepsilon}, X)$, $\gamma x \in B_p(\frac{1}{\varepsilon}, X)$, then $d(f(\gamma x), \varphi(\gamma)(f(x))) < \varphi$;

(S5) if $\mu \in \Lambda(\frac{1}{\varepsilon})$, $x \in B_p(\frac{1}{\varepsilon}, X)$, $\psi(\mu)(x) \in B_p(\frac{1}{\varepsilon}, X)$, then $d(f(\psi(\mu)(x)), \mu f(x)) < \varepsilon$.

The *equivariant pointed Gromov–Hausdorff distance* $d_{e\mathrm{GH}}((X, \Gamma, p), (Y, \Lambda, q))$ stands for the infimum of the positive numbers ε such that there exist ε-equivariant pointed Hausdorff approximations from (X, Γ, p) to (Y, Λ, q) and from $(Y, \Lambda, q))$ to (X, Γ, p).

Proof of Theorem 1.1. Let ω_k be a sequence of Kähler metrics on M in the same Kähler class with Ricci curvature $\geqslant -\frac{1}{k}\omega$. Let $\widetilde{M}_k$ be the Riemannian covering space of M_k (the manifold M with the Kähler metric ω_k). Using Lemma 2.1 Paun [12] proved that there is an open subset $\widetilde{U}_k \subset \widetilde{M}_k$ of $\mathrm{diam}_{\omega_k}(\widetilde{U}_k) \leqslant C$ such that the homomorphism $\pi_1(U_k) \to \pi_1(M_k)$ is surjective, where U_k is the image of $\widetilde{U}_k$ in M_k, C is a universal constant independent of k.

For convenience let $G = \pi_1(M)$, and let $\Gamma = G/[G, G]$. Consider $\overline{M}_k = \widetilde{M}_k/[G, G]$. By assumption $\mathbb{Z}^{2n} \subset \Gamma$ acts on $\overline{M}_k$ by isometry. By a lemma of Gromov [9] (compare [12]) it follows that there is a finite index torsion free subgroup Γ_k of Γ such that, fixing a base point $p_k \in \overline{U}_k \subset \overline{M}_k$,

(S6) the geometric norm of any non-trivial element of Γ_k is at least C,

(S7) Γ_k is generated by $\gamma_1, \ldots, \gamma_{2n}$ so that the geometric norm of every γ_i is at most $2C$.

Since Γ_k acts on $\overline{M}_k$ by isometry, the quotient space $\overline{M}_k/\Gamma_k$ is a finite Riemannian covering space of M_k. Because the Ricci curvature of $\overline{M}_k$ is bounded from below, by the Gromov compactness theorem (cf. [7]) the pointed spaces converge

$$\left(\overline{M}_k, \Gamma_k, p_k\right) \xrightarrow{d_{e\mathrm{GH}}} (X, \Gamma_\infty, q)$$

in the equivariant Gromov–Hausdorff topology when k tends to infinity. By (S6) it is easy to see that the isometric action of Γ_∞ on X is discrete and effective. By the splitting theorem [3] the limit space $X = Y \times \mathbb{R}^\ell$, where Y contains no line. By [9] it is well known the Hausdorff dimension of X is at most $2n$, therefore $\ell \leqslant 2n$. We first need

Lemma 2.3. $\Gamma_\infty \cong \mathbb{Z}^{2n}$.

Proof of Lemma 2.3. By definition, there are maps $\varphi_k : \Gamma_k(k) \to \Gamma_\infty(k)$, $\psi_k : \Gamma_\infty(k) \to \Gamma_k(k)$ and a $\frac{1}{k}$-Hausdorff approximation $f_k : B_{p_k}(k, \overline{M}_k) \to B_q(k, X)$ satisfying (S1)–(S5).

We first claim that φ_k is injective for sufficiently large k. If not, there are two elements $\gamma_k \neq \lambda_k \in \Gamma_k(k)$ such that $\varphi(\gamma_k) = \varphi(\lambda_k)$ for any k. Let $\mu_k = \varphi(\gamma_k) = \varphi(\lambda_k)$. Put $x = p_k$. By (S4) we get that $d(f_k(\lambda_k x), \mu_k f_k(x)) < \frac{1}{k}$ and $d(f_k(\gamma_k x), \mu_k f_k(x)) < \frac{1}{k}$. Therefore, $d(f_k(\lambda_k x), f_k(\gamma_k x)) < \frac{2}{k}$ and so $d(\lambda_k \gamma_k^{-1} x, x) = d(\lambda_k x, \gamma_k x) < \frac{4}{k}$ since f_k is a $\frac{1}{k}$-Hausdorff approximation. A contradiction to (S6).

Secondly, we claim that $\varphi_k(\gamma_i\gamma_j) = \varphi_k(\gamma_i)\varphi_k(\gamma_j) = \varphi_k(\gamma_j)\varphi_k(\gamma_i)$ for any $\gamma_i, \gamma_j \in \Gamma_k(k)$ so that $\gamma_i\gamma_j \in \Gamma_k(k)$. In fact, by (S4) again we get that $d(\varphi_k(\gamma_i\gamma_j) f_k(x), f_k(\gamma_i\gamma_j x)) < \frac{1}{k}$; $d(\varphi_k(\gamma_i)\varphi_k(\gamma_j) f_k(x), \varphi_k(\gamma_i) f_k(\gamma_j x)) < \frac{1}{k}$ and $d(f_k(\gamma_i\gamma_j x), \varphi_k(\gamma_i) f_k(\gamma_j x)) < \frac{1}{k}$. Thus, $d(\varphi_k(\gamma_i\gamma_j) f_k(x), \varphi_k(\gamma_i)\varphi_k(\gamma_j) f_k(x)) < \frac{3}{k}$. For the same reason as above, by (S6) it follows that $\varphi_k(\gamma_i\gamma_j) = \varphi_k(\gamma_i)\varphi_k(\gamma_j)$. The claim follows.

Similar argument applies to show that $\varphi_k(\gamma_i^{-1}) = \varphi_k(\gamma_i)^{-1}$, if $\gamma_i, \gamma_i^{-1} \in \Gamma_k(k)$.

Next we verify that $\varphi_k : \Gamma_k(k) \to \Gamma_\infty(k)$ is also surjective.

We argue by contradiction. Assume such an element $\mu_k \in \Gamma_\infty(k)$. By (S5) $d(f_k(\psi(\mu_k)(x), \mu_k f_k(x)) < \frac{1}{k}$. By (S4) $d(f_k(\psi(\mu_k)(x), \varphi_k(\psi(\mu_k)) f_k(x)) < \frac{1}{k}$. Therefore, $d(\varphi_k(\psi(\mu_k)) f_k(x), \mu_k f_k(x)) < \frac{2}{k}$. By (S6) this implies that $\mu_k = \varphi_k(\psi(\mu_k))$. A contradiction.

For sufficiently large k, let Γ_0 be the subgroup of Γ_∞ generated by $\varphi_k(\gamma_1), \dots, \varphi_k(\gamma_{2n})$. It may be verified easily that this does not depend on the choice of k. By (S7) and the above Γ_0 is a commutative group of rank $2n$. Since φ_k is surjective, $\Gamma_0 = \Gamma_\infty$. The desired result follows. □

To continue the proof of Theorem 1.1, we first prove that $X = \mathbb{R}^{2n}$. It suffices to show that $\ell = 2n$.

We argue by contradiction. Assume $\ell < 2n$.

Since Γ_∞ preserves the splitting, there is a well-defined homomorphism $p : \Gamma_\infty \to \mathrm{Isom}(\mathbb{R}^\ell)$. Let $\Gamma_{0,\infty}$ denote the kernel of p. By the generalized Bieberbach theorem (cf. [7]) the image $p(\Gamma_\infty)$ has rank at most ℓ. By Lemma 2.3 $\Gamma_{0,\infty}$ has rank $\geqslant 1$. For a non-trivial element of $\mu \in \Gamma_{0,\infty}$, by (S5) there is a sequence of element $\gamma_k = \psi_k(\mu) \in \Gamma_k$ (of infinite order) such that the γ_k-action on $\overline{M}_k$ converges to the action of μ on $Y \times \mathbb{R}^\ell$. Observe that a minimal geodesic representation in $\overline{M}_k/\Gamma_k$ gives rise a line in $\overline{M}_k$, on which γ_k acts by deck transformation. This sequence of lines converges to a line in Y on which μ acts by translation. Therefore the line lies in Y since $\mu \in \Gamma_{0,\infty}$ acts trivially on the factor $\mathbb{R}^\ell$. A contradiction to the assumption that Y has no line. Hence $\ell = 2n$.

Finally, by (S7) we see that $\mathbb{R}^{2n}/\Gamma_\infty$ is compact. By [7] Lemma 3.4 $\overline{M}_k/\Gamma_k$ converges to $\mathbb{R}^{2n}/\Gamma_\infty$. This shows that $\overline{M}_k/\Gamma_k$ has uniformly bounded diameter. Therefore, $\overline{M}_k/\Gamma_k$ has almost non-negative Ricci curvature in Gromov's sense [9]. By [4] we conclude that $\overline{M}_k/\Gamma_k$ is homeomorphic to a torus T^{2n}, and so is M. By Poincaré–Lelong equation it follows that the Albanese map has no zeros and is actually a biholomorphism. This completes the proof of Theorem 1.1. □

Remark 2. The above proof actually shows that a sequence of Kähler metrics on T^{2n} in the same Kähler class $[\omega]$ with Ricci curvature $\geqslant -\varepsilon\omega$ has uniformly bounded diameter, and so the metrics do not collapse.

Let $G = \pi_1(M)$. Consider the lower central series

$$\cdots G_2 \subset G_1 \subset G_0 = G,$$

where $G_1 = [G, G]$ and $G_2 = [G_1, G]$. Let $G_2' \subset G$ be the normal subgroup such that $G/G_2' = (G/G_2)/\text{torsion}$. Assume $H_1(G)/\text{torsion} \cong \mathbb{Z}^{2n-2}$, and $\mathrm{rank}(G/G_2') = 2n - 2 + m$. By [12] we may assume elements $\gamma_1, \dots, \gamma_{2n-2}$; $\alpha_1, \dots, \alpha_m \in G$ which generate a finite index subgroup $\Gamma_k' \subset G/G_2'$ and satisfy (S6), (S7) and

(S8) the geometric norms of $\alpha_1, \dots, \alpha_m$ are all less than $2C$.

We warn that this is not true if we require that $\alpha_1, \dots, \alpha_m$ satisfy (S7).

Now we start the proof of Theorem 1.2. We will only sketch the main steps since the proof follows the same line as the previous one.

Proof of Theorem 1.2. Let $\overline{M}_k' = \widetilde{M}_k/G_2'$. Consider the triple $(\overline{M}_k', \Gamma_k', p_k)$. The pointed spaces converge

$$\left(\overline{M}_k', \Gamma_k', p_k\right) \xrightarrow{d_{e\mathrm{GH}}} (X, \Gamma_\infty', q)$$

Exactly the same argument in the previous proof implies that $X = Y \times \mathbb{R}^{2n-2}$ and Y contains at least a line since the group generated by $\{\alpha_1, \dots, \alpha_m\}$ converges to a non-trivial isometry group acting on X acting trivially on $\mathbb{R}^{2n-2}$, where Y is a length space of Hausdorff dimension at most two. However, since (S7) is not satisfied for the α_i's, the limit group Γ_∞' may not be discrete (compare [7] Example 3.11). Therefore, $X = Y_0 \times \mathbb{R}^{2n-1}$ where the Hausdorff dimension of Y_0 is at most 1.

If Y_0 is compact, e.g., zero-dimensional, by (S7) and (S8) it follows that the limit space $Y_0 \times \mathbb{R}^{2n-1}/\Gamma'_\infty$ is compact. Therefore, the diameters of the sequence $\overline{M}'_k/\Gamma'_k$ have a uniform upper bound, so are the diameters of M_k (since M_k is a finite isometric quotient of $\overline{M}'_k/\Gamma'_k$). By [13] it follows that the Albanese map is surjective.

If Y_0 is 1-dimensional and non-compact, clearly, Y_0 has two ends and thus Y_0 contains a line. By [3] once again $Y_0 = \mathbb{R}$. This proves that $X \cong \mathbb{R}^{2n}$. Since $m \geqslant 2$, the rank of Γ'_∞ is at least $2n$ (may be non-discrete). By the generalized Bieberbach theorem the quotient $\mathbb{R}^{2n}/\Gamma'_\infty$ has to be compact. For the same reasoning as in the proof of Theorem 1.1 the desired result follows. □

Acknowledgements

The author would like to thank Ngaiming Mok for helpful discussions concerning the Poincaré–Lelong equation. The Note was written during the author's visit to the Max-Planck Institute für Mathematik and IHES. The author is very grateful to these institutes for financial support. Finally, the author would like to thank Dr. Xiangdong Li for the French translation of the introduction.

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