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Analytic Geometry

Approximation of analytic sets with proper projection by Nash sets

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Abstract

Let X be an analytic subset of $U \times \mathbf{C}^k$ of pure dimension such that the projection $\pi : X \to U$ is surjective and proper, where U is a Runge domain. We show that X can be approximated by Nash sets. ***To cite this article: M. Bilski, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Approximation des ensembles analytiques à projection propre par des ensembles de Nash. Soit $X \subset U \times \mathbf{C}^k$ un ensemble analytique de dimension pure tel que la projection $\pi : X \to U$ est surjective et propre, où U est un domaine de Runge. Nous démontrons que X est approchable par des ensembles de Nash. ***Pour citer cet article : M. Bilski, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction and preliminaries

The problem of algebraic approximation of analytic objects appears naturally in complex geometry and has been considered by several mathematicians (see [5,7–10]). Various results have been obtained, especially when approximated objects are holomorphic mappings or manifolds. This note is devoted to the problem of approximating analytic sets by Nash sets. The approximation is expressed by means of the *convergence of holomorphic chains* (which, in the context of our interest, is equivalent to the *convergence of the currents of integration* associated with the considered sets).

First let us introduce the notion of *local uniform convergence* which together with some additional conditions constitutes the definition of the convergence of chains. Let U be an open subset of $\mathbf{C}^n$ and let Y, Y_ν, for $\nu \in \mathbf{N}$, be closed subsets of U. We say that $\{Y_\nu\}$ converges to Y locally uniformly if and only if:

(1) for every $a \in Y$ there is a sequence $\{a_\nu\}$ with $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^n$,
(2) for every compact subset K of U such that $K \cap Y = \emptyset$ it holds $Y_\nu \cap K = \emptyset$ for almost all ν.

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Then we write $Y_\nu \to Y$. For details concerning the topology of local uniform convergence see [13], where approximation of set-theoretic complete intersections in this topology is also discussed. Some special non-complete intersection cases are handled in [2].

Let us present the main results of this note. Assuming the notation of Subsection 1.2 and taking into account that every analytic subset of an open set in $\mathbf{C}^m$ can be considered as a holomorphic chain such that each of its components has multiplicity 1, we prove

Theorem 1.1. *Let U be a connected Runge domain in $\mathbf{C}^n$ and let X be an analytic subset of $U \times \mathbf{C}^k$ of pure dimension n with proper projection onto U. Then for every open relatively compact subset V of U there is a sequence of Nash subsets of $V \times \mathbf{C}^k$ of pure dimension n with proper projection onto V, converging to $X \cap (V \times \mathbf{C}^k)$ in the sense of holomorphic chains.*

In view of the local structure of analytic sets (see [14]) we directly have the following

Corollary 1.2. *Let Ω be an open subset of $\mathbf{C}^m$, $m \in \mathbf{N}$ and let X be an analytic subset of Ω of pure dimension n. Then for every $a \in X$ there exist an open neighborhood U of a in Ω and a sequence of Nash subsets of U of pure dimension n converging to $X \cap U$ in the sense of holomorphic chains.*

These results can be equivalently formulated in terms of the convergence of currents of integration over analytic sets in the weak-$\star$ topology (introduced in [6], see also [4]). (For equivalence, in the considered context, see [3], pp. 141, 206, 207.) The crucial tool which is used in the proof of our results is Theorem 2.1 which is due to Lempert (see [7]). Let us briefly present preliminary material.

1.1. Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a *Nash function* at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a *Nash function* if it is a Nash function at every point of Ω.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a *Nash subset* of Ω if and only if for every $y_0 \in \Omega$ there exist a neighborhood of y_0 in which Y is defined by a finite number of Nash equations (cp. [11]).

We will use the following fact from [11] (p. 239). Let $\pi : \Omega \times \mathbf{C}^k \to \Omega$ denote a natural projection.

Theorem 1.3. *Let X be a Nash subset of $\Omega \times \mathbf{C}^k$ such that $\pi|_X : X \to \Omega$ is a proper mapping. Then $\pi(X)$ is a Nash subset of Ω and* $\dim(X) = \dim(\pi(X))$.

1.2. Holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean the formal sum $A = \sum_{j\in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j\in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [12], cp. also [1,3]). The set $\bigcup_{j\in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an n-chain.

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1) $|Z_\nu| \to |Z|$,
(2) for each regular point a of $|Z|$ and each submanifold T of U of dimension $m - n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact and $|Z| \cap \overline{T} = \{a\}$, we have $\deg(Z_\nu \cdot T) = \deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \rightarrowtail Z$. (By $Z \cdot T$ we denote the intersection product of Z and T (cf. [12]). Observe that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^d \alpha_j \{a_j\}$, $\deg(A) = \sum_{j=1}^d \alpha_j$).

The following lemma from [12], p. 181, will be useful to us.

Lemma 1.4. *Let $n \in \mathbf{N}$ and Z, Z_ν, for $\nu \in \mathbf{N}$, be positive n-chains. If $|Z_\nu| \to |Z|$ then the following conditions are equivalent*:

(1) $Z_\nu \rightarrowtail Z$,
(2) *for each point a from a given dense subset of* $\operatorname{Reg}(|Z|)$ *there exists a submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact, $|Z| \cap \overline{T} = \{a\}$ and* $\deg(Z_\nu \cdot T) = \deg(Z \cdot T)$ *for almost all ν.*

1.3. Symmetric powers

Let $(\mathbf{C}^k)^d_{\mathrm{sym}}$ and $\langle x_1, \ldots, x_d\rangle$ denote $(\mathbf{C}^k)^d/_\sim$ and the class of abstraction of $(x_1, \ldots, x_d) \in (\mathbf{C}^k)^d$ respectively, where $(x'_1, \ldots, x'_d) \sim (x_1, \ldots, x_d)$ iff $(x'_1, \ldots, x'_d) = (x_{p(1)}, \ldots, x_{p(d)})$, for some permutation p. We endow $(\mathbf{C}^k)^d_{\mathrm{sym}}$ with a metric d given by $d(\langle x_1, \ldots, x_d\rangle, \langle y_1, \ldots, y_d\rangle) = \inf_p \sup_i \|x_i - y_{p(i)}\|_{\mathbf{C}^k}$, where p is any permutation of $(1, \ldots, d)$.

Then there exist an integer N and a mapping $\phi : (\mathbf{C}^k)^d_{\mathrm{sym}} \to \mathbf{C}^N$ with the following properties (cf. [14], pp. 366–368, 152–154):

(a) ϕ is injective and ϕ, ϕ^{-1} are continuous and proper,
(b) $\phi \circ \pi_{\mathrm{sym}} : (\mathbf{C}^k)^d \to \mathbf{C}^N$ is a polynomial mapping, where $\pi_{\mathrm{sym}}(x_1, \ldots, x_d) = \langle x_1, \ldots, x_d\rangle$,
(c) $\phi((\mathbf{C}^k)^d_{\mathrm{sym}})$ is an algebraic subset of $\mathbf{C}^N$. (In [14] the analyticity of $\phi((\mathbf{C}^k)^d_{\mathrm{sym}})$ is proved. By this proof and Theorem 1.3, $\phi((\mathbf{C}^k)^d_{\mathrm{sym}})$ is a Nash and irreducible subset of $\mathbf{C}^N$, thus algebraic (cf. [11], p. 237).)

Note. Such a map ϕ can be obtained by taking $\phi \circ \pi_{\mathrm{sym}}$ equal to the collection of elementary symmetric functions $(\mathbf{C}^k)^d \to \bigoplus_{1 \leqslant p \leqslant d} S^p(\mathbf{C}^k)$, $(x_1, \ldots, x_d) \mapsto \bigoplus_{1 \leqslant p \leqslant d} \sum_{j_1 < \cdots < j_p} x_{j_1} \cdots x_{j_p}$ into the symmetric algebra of $\mathbf{C}^k$ (identifying the vector space $\bigoplus_{1 \leqslant p \leqslant d} S^p(\mathbf{C}^k)$ with $\mathbf{C}^N$ for some N).

2. Proof of Theorem 1.1

We shall use the following theorem which is due to Lempert (see [7] Theorem 3.2, pp. 338, 339).

Theorem 2.1. *Let K be a holomorphically convex compact subset of $\mathbf{C}^n$ and $f : K \to \mathbf{C}^k$ a holomorphic mapping that satisfies a system of equations $Q(z, f(z)) = 0$ for $z \in K$. Here Q is a Nash mapping from a neighborhood $U \subset \mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then f can be uniformly approximated by a Nash map $F : K \to \mathbf{C}^k$ satisfying $Q(z, F(z)) = 0$.*

Proof of Theorem 1.1. Put $d := \max\{\sharp((\{x\} \times \mathbf{C}^k) \cap X)\colon x \in U\}$. Let Σ denote the set $\{x \in U\colon \sharp((\{x\} \times \mathbf{C}^k) \cap X) < d\}$ which is a proper analytic subset of U (see [14]). Next, for every x in $U \setminus \Sigma$, let $a_1(x), \ldots, a_d(x) \in \mathbf{C}^k$ be the points from $(\{x\} \times \mathbf{C}^k) \cap X$. Recall that for every $x \in U \setminus \Sigma$ there is a neighborhood in U in which $a_1, \ldots, a_d$ are holomorphic mappings (see [14]). Next recall that there are an integer N and a mapping $\phi : (\mathbf{C}^k)^d_{\mathrm{sym}} \to \mathbf{C}^N$ as in Section 1.3. Then the mapping $\psi : U \setminus \Sigma \to \mathbf{C}^N$ given by the formula $\psi(x) = \phi(\langle a_1(x), \ldots, a_d(x)\rangle)$ is holomorphic and by the Riemann theorem can be extended to a holomorphic mapping $\tilde{\psi} : U \to \mathbf{C}^N$.

By the properties of ϕ, the set $\phi((\mathbf{C}^k)^d_{\mathrm{sym}})$, which contains $\tilde{\psi}(U)$, is an algebraic subset of $\mathbf{C}^N$. Now by Theorem 2.1, in view of the fact that U is a Runge domain, we have the following. For every open $V \subset\subset U$ there exists a sequence of Nash mappings $\psi_\nu : V \to \mathbf{C}^N$ such that ψ_ν converges to $\tilde{\psi}|_V$ uniformly and $\psi_\nu(V) \subset \phi((\mathbf{C}^k)^d_{\mathrm{sym}})$ for every ν.

Using the mapping ψ_ν we define the set $X_\nu \subset V \times \mathbf{C}^k$ as follows:

$$X_\nu = \left\{(x, z) \in V \times \mathbf{C}^k \mid \exists z_2, \ldots, z_d \in \mathbf{C}^k\colon\ (\phi \circ \pi_{\mathrm{sym}})(z, z_2, \ldots, z_d) = \psi_\nu(x)\right\}.$$

The properties of ϕ and the fact that $\psi_\nu(V) \subset \phi((\mathbf{C}^k)^d_{\mathrm{sym}})$ and ψ_ν is a Nash mapping imply that

$$\widetilde{X}_\nu = \left\{(x, z, z_2, \ldots, z_d) \in V \times \left(\mathbf{C}^k\right)^d\colon\ (\phi \circ \pi_{\mathrm{sym}})(z, z_2, \ldots, z_d) = \psi_\nu(x)\right\}$$

is a Nash set of pure dimension n. Thus X_ν is also such because it is the image of $\widetilde{X}_\nu$ by a proper projection (see Theorem 1.3).

Finally, the property (a) of ϕ easily implies that $\{X_\nu\}$ converges to $X \cap (V \times \mathbf{C}^k)$ locally uniformly and moreover, for every ν it holds: $\max\{\sharp((\{x\} \times \mathbf{C}^k) \cap X_\nu)\colon\ x \in V\} = d$. Thus by Lemma 1.4 we obtain the convergence in the sense of chains. □

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