

Numerical Analysis

# Gradient-prolongation commutativity and graph theory

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## Abstract

This Note gives conditions that must be imposed to algebraic multilevel discretizations involving at the same time nodal and edge elements so that a gradient-prolongation commutativity condition will be satisfied; this condition is very important, since it characterizes the gradients of coarse nodal functions in the coarse edge function space. They will be expressed using graph theory and they provide techniques to compute approximation bases at each level. *To cite this article: F. Musy et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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## Résumé

**Commutativité entre gradient et prolongement et théorie des graphes.** Cette Note donne des conditions qui doivent être imposées aux discrétisations multiniveau algébriques en éléments finis nodaux et d'arête de façon à assurer la commutativité entre gradient et prolongement; cette relation importante caractérise les gradients des fonctions nodales grossières dans l'espace des fonctions d'arête grossières. Ces conditions seront exprimées en terme de graphes et elles permettent d'introduire des méthodes de calcul des bases d'approximation aux différents niveaux. *Pour citer cet article: F. Musy et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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## Version française abrégée

L'approximation numérique du champ électrique ou magnétique utilise fréquemment les éléments finis d'arête dont la relation avec les éléments finis nodaux traduit des propriétés importantes au niveau discret [1]. Dans ce qui suit, nous considérerons les éléments de plus bas degré :  $P_1$  en nodal et ordre 1 incomplet pour les arêtes. Dès qu'on traite des problèmes de grande taille, une stratégie multiniveau est un choix intéressant. Pour les systèmes provenant de discrétisations par éléments finis d'arête, Hiptmair a introduit des méthodes multiniveau pour une hiérarchie de maillages emboîtés [2].

Cependant, dans des applications réalistes, on ne dispose généralement pas de maillages structurés. La stratégie multiniveau algébrique va donc s'imposer : il s'agit de définir des fonctions grossières nodales et d'arête grâce aux

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contributions de paquets de fonctions fines nodales et d'arête; les combinaisons linéaires (1a) et (1b) définissent respectivement ces fonctions grossières nodales et d'arête.

Par construction les gradients des fonctions nodales fines appartiennent à l'espace des fonctions d'arête fines ce que traduit la relation (2). Dans cette relation,  $G^h$  est la matrice d'incidence arcs-sommets du graphe orienté naturellement associé au maillage de travail. Les orientations des arcs sont arbitraires.

Pour adapter aux méthodes algébriques les lisseurs des méthodes géométriques, Reitzinger et Schöberl [5] ont introduit une représentation explicite des gradients des fonctions grossières nodales dans la base des fonctions grossières d'arête, donnée par la relation (3) où  $G^H$  est une matrice d'incidence arcs-sommets.

En regroupant les relations (1) à (3), nous obtenons la relation matricielle (4). La matrice  $\alpha$  est construite par exemple par les méthodes définies dans [3] qui permettent d'obtenir les fonctions grossières nodales comme partition de l'unité et de contraindre leurs supports à être inclus dans des ensembles géométriques convenablement choisis.

Connaissant  $G^h$  et  $\alpha$ , nous souhaitons choisir  $G^H$  comme matrice d'incidence arcs-sommets d'un graphe orienté  $\mathcal{S}^H$ . Nous donnons dans cette note une condition nécessaire et suffisante sur ce graphe, la Proposition 2.3, qui assure l'existence d'une solution de (4). En effet, nous associerons, par un procédé décrit dans la partie en anglais, à chaque arête fine un sous-graphe du graphe grossier, qui doit être connexe.

La connaissance de ces sous-graphes donne les degrés de liberté disponibles pour déterminer des fonctions d'arête grossières compatibles avec les fonctions nodales grossières; en résolvant un problème de flot sur ces sous-graphes, voir (14), nous pouvons alors construire la matrice  $\beta$  (Section 4).

## 1. Introduction

Numerical approximation of electric or magnetic field uses often edge finite elements whose relation with nodal finite elements contains important properties at discrete level [1]. In this Note we restrict ourselves to lowest order approximation:  $P_1$  for nodal elements and incomplete order 1 for edge elements. In order to solve large problems, multilevel methods are an attractive choice. While, for systems coming from edge element discretization, Hiptmair [2] proposed multilevel methods using nested meshes, engineering applications do not usually provide structured meshes. Therefore, algebraic multilevel methods are an interesting option: we have to build coarse nodal and edge functions by using aggregates of fine nodal and edge functions. If  $(\phi_p^h)_{p=1,\dots,N^h}$  and  $(\lambda_i^h)_{i=1,\dots,E^h}$  respectively denote fine nodal and edge bases, the following linear combinations define coarse nodal and edge functions:

$$\phi_n^H = \sum_{p=1}^{N^h} \alpha_{pn} \phi_p^h, \quad \forall n \in \{1, \dots, N^H\}, \quad (1a)$$

$$\lambda_e^H = \sum_{i=1}^{E^h} \beta_{ie} \lambda_i^h, \quad \forall e \in \{1, \dots, E^H\}. \quad (1b)$$

By construction, the gradients of fine nodal functions belong to the space of fine edge functions:

$$\forall p \in \{1, \dots, N^h\}, \quad \text{grad}(\phi_p^h) = \sum_{i=1}^{E^h} G_{ip}^h \lambda_i^h, \quad (2)$$

where  $G^h$  is the edge-node incidence matrix of the digraph naturally associated with the initial mesh. The orientation of the edges can be arbitrarily chosen.

In [5], Reitzinger and Schöberl deduced their smoother from the matrix  $G^H$  involved in the relation:

$$\forall n \in \{1, \dots, N^H\}, \quad \text{grad}(\phi_n^H) = \sum_{e=1}^{E^H} G_{en}^H \lambda_e^H, \quad (3)$$

which states that the gradients of the coarse nodal functions must also belong to the space of coarse edge functions. The matrix  $G^H$  is an edge-node incidence matrix as in the structured case. Relation (3) does not guarantee the efficacy of the algebraic multilevel method but it leads to relevant strategies.

Gathering Eqs. (1), (2) and (3), we obtain the matrix relation:

$$G^h \alpha = \beta G^H. \tag{4}$$

The matrix  $\alpha$  is constructed following for instance the methods defined in [3], which provides a family of coarse nodal functions, making up a partition of unity, whose supports satisfy appropriate conditions.

Knowing the left-hand side of (4), we want to choose  $G^H$  as an edge-node incidence matrix of a digraph  $\mathcal{S}^H$ , and we will give conditions on the coarse graph  $\mathcal{S}^H$ , which ensure the existence of a matrix  $\beta$  satisfying (4). Moreover, the proof of the proposition indicates how to choose the degrees of freedom which enables us to define the coarse edge functions. It also helps us to construct  $\beta$ .

## 2. Notation and statement of the problem

Let  $(L_n)_{n=1,\dots,N^H}$  be sets of indices in  $\{1, \dots, N^h\}$  such that:

$$\bigcup_{n=1}^{N^H} L_n = \{1, \dots, N^h\}. \tag{5}$$

The matrix  $\alpha$  describes the coarse nodal basis; we assume that it has been previously computed and it has the following properties:

- the coarse nodal functions make up a partition of unity, which can be algebraically stated as:

$$\forall p \in \{1, \dots, N^h\}, \quad \sum_{n=1}^{N^H} \alpha_{pn} = 1, \tag{6}$$

- in order to restrict the support of each coarse basis function  $\phi_n^H$ , the indices of the non-zero components of  $\phi_n^H$  are included in the set  $L_n$ , i.e.:

$$p \in \{1, \dots, N^h\} \setminus L_n \Rightarrow \alpha_{pn} = 0. \tag{7}$$

The fine nodal function  $\phi_p^h$  contributes to the coarse nodal function  $\phi_n^H$  if  $p$  belongs to  $L_n$ .

We have a reciprocal set-valued function  $\tilde{L}$ : the set  $\tilde{L}_p$  is the set of coarse nodal function indices to which the fine nodal function  $\phi_p^h$  contributes. For the fine graph in Fig. 1(a), we set  $L_1 = \{1, 2, 3, 4, 5, 6, 7\}$ ,  $L_2 = \{5, 6, 8, 9, 13, 14\}$  and  $L_3 = \{7, 8, 10, 11, 12\}$ . One obtains, for instance, the set  $\tilde{L}_7 = \{1, 3\}$ .

We define two families of sets of fine edge function indices. We will denote a directed fine edge  $i$  by  $\overline{pq}^h$  where  $p$  and  $q$  are respectively the starting and ending nodes of the edge  $i$ . A similar notation is used for a directed coarse edge  $e = \overline{mn}^H$ .

The set  $C_n$  is the set of indices of fine edges which have an extremity in  $L_n$ :

$$C_n = \{i \in \{1, \dots, E^h\} : i = \overline{pq}^h, p \in L_n \text{ or } q \in L_n\}. \tag{8}$$

The fine edge function  $\lambda_i^h$  contributes to the gradient of the coarse nodal function  $\phi_n^H$  if  $i$  belongs to  $C_n$ . Indeed, for the directed fine edge  $i = \overline{pq}^h$ ,  $G_{ir}^h$  is equal to  $-1$  if  $r = p$  and  $+1$  if  $r = q$ . Moreover, if  $p$  and  $q$  are not in  $L_n$ , the components  $\alpha_{pn}$  and  $\alpha_{qn}$  vanish according to (7); therefore:

$$i \in \{1, \dots, E^h\} \setminus C_n \Rightarrow (G^h \alpha_{\bullet n})_i = 0, \tag{9}$$

where  $\alpha_{\bullet n}$  denotes the  $n$ -th column of  $\alpha$ . The reciprocal set-valued function  $\tilde{C}$  is such that  $\tilde{C}_i$  is the set of coarse nodal function indices to whose gradient the fine edge function  $\lambda_i^h$  contributes. On Fig. 1(b), the fine edges are numbered, set  $C_3$  is highlighted and we can note, for instance, the set  $\tilde{C}_8 = \{1, 3\}$ .

Let  $e = \overline{mn}^H$  be an edge of the coarse graph  $\mathcal{S}^H$ ; we define:

$$I_e = C_n \cap C_m. \tag{10}$$

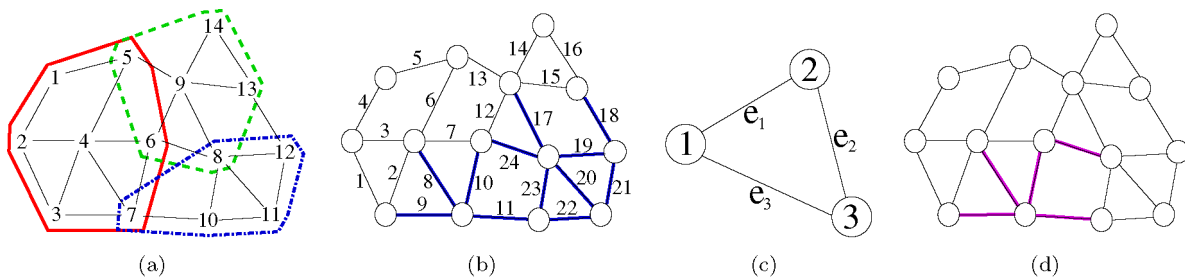


Fig. 1. Representation of the fine and coarse graphs, sets  $(L_n)_{n=1,\dots,3}$ ,  $C_3$  and  $I_{e_3}$ : (a) fine graph with node indices. Sets  $(L_n)_{n=1,\dots,3}$  are surrounded; (b) fine graph with edge indices. Set  $C_3$  is represented by bold edges; (c) coarse graph; (d) set  $I_{e_3}$  is represented by bold edges.

By analogy with the structured case and for restricting the support of  $\lambda_e^H$ , we enforce:

$$i \in \{1, \dots, E^h\} \setminus I_e \Rightarrow \beta_{ie} = 0. \quad (11)$$

The fine edge function  $\lambda_i^h$  contributes to the coarse edge function  $\lambda_e^H$  if  $i$  belongs to  $I_e$ . The set-valued function  $\tilde{I}$  is such that  $\tilde{I}_i$  is the set of coarse edge function indices to which the fine edge function  $\lambda_i^h$  contributes. The coarse graph in Fig. 1(c) is related to the fine in Fig. 1(a). Set  $I_{e_3}$  is represented in Fig. 1(d).

The following statement can be easily deduced from (8) and the definition of  $G^h$ :

**Lemma 2.1.** *If  $i$  denotes the edge  $\overline{pq}^h$ ,  $\tilde{C}_i = \tilde{L}_p \cup \tilde{L}_q$ .*

In order to simplify notations, we introduce the set  $\tilde{F} = \{i \in \{1, \dots, E^h\}; \tilde{I}_i \neq \emptyset\}$ , since some fine edge functions might not contribute to any coarse edge functions.

For any fine edge  $i$ , let  $\mathcal{S}^{H,i}$  be the induced subgraph defined by  $\tilde{C}_i$ : the vertices of  $\mathcal{S}^{H,i}$  are the vertices of  $\mathcal{S}^H$ , which are indexed by the elements of  $\tilde{C}_i$  and the edges of  $\mathcal{S}^{H,i}$  are those edges of  $\mathcal{S}^H$  whose extremities are vertices of  $\mathcal{S}^{H,i}$ .

The following lemma is a direct consequence of definition (10):

**Lemma 2.2.** *For any edge  $i \in \tilde{F}$ , the edges of  $\mathcal{S}^{H,i}$  are those edges of  $\mathcal{S}^H$  which are indexed by  $\tilde{I}_i$ .*

We may now state precisely our main result, which gives a necessary and sufficient condition on the coarse graph  $\mathcal{S}^H$  permitting the resolution of (4):

**Proposition 2.3.** *For all matrices  $\alpha$  satisfying conditions (6) and (7), there exists a matrix  $\beta$  satisfying (11) and solving (4) iff for all  $i$ , the induced subgraph  $\mathcal{S}^{H,i}$  is connected.*

### 3. The essential steps of the proof

**First step.** Many relations in (4) reduce to  $0 = 0$ : this is the case for  $n \notin \tilde{C}_i$ .

Indeed, according to (9) the  $(i, n)$  coefficient of the right-hand side of (4) vanishes.

Conversely, if  $e$  does not belong to  $\tilde{I}_i$ , according to (11) and the definition of  $\tilde{I}_i$ ,  $\beta_{ie}$  vanishes and:

$$\sum_{e=1}^{E^H} \beta_{ie} G_{en}^H = \sum_{e \in \tilde{I}_i} \beta_{ie} G_{en}^H. \quad (12)$$

On the other hand if the directed coarse edge  $e$  denoted by  $\overline{lm}^H$  belongs to  $\tilde{I}_i$ , Lemma 2.2 implies that  $l$  and  $m$  belongs to  $\tilde{C}_i$ . However, for  $G_{en}^H$  not to vanish for all  $e$ ,  $m$  or  $l$  must be equal to  $n$ , which means that  $n$  belongs to  $\tilde{C}_i$ , and this contradicts the assumption  $n \notin \tilde{C}_i$ .

**Second step.** We look at all the other equations, i.e. those for which  $n \in \tilde{C}_i$ . We note that (12) remains and that the edges indexed by  $\tilde{I}_i$  are precisely those of the graph  $\mathcal{S}^{H,i}$  according to Lemma 2.2.

We assume now  $i \in \tilde{F}$  and we define  $G^{H,i}$  as the edge-node incidence matrix of  $\mathcal{S}^{H,i}$  and the  $(i, n)$  equation of (4) is rewritten:

$$\sum_{e \in \tilde{I}_i} \beta_{ie} G_{en}^{H,i} = \Theta_{i,n} \quad \text{where } \Theta_{i,n} = \sum_{r \in L_n} G_{ir}^h \alpha_{rn}. \tag{13}$$

This could be satisfied for all couples  $(i, n)$  such that  $n \in \{1, \dots, N^H\}$  and  $i \in C_n$  or equivalently  $i \in \{1, \dots, E^h\}$  and  $n \in \tilde{C}_i$ . For a fixed  $i$ , we may write that  $\beta_{i\bullet}$ , the  $i$ -th row of  $\beta$  satisfies the system:

$$\sum_{e \in \tilde{I}_i} \beta_{ie} G_{en}^{H,i} = \Theta_{i,n}, \quad \forall n \in \tilde{C}_i. \tag{14}$$

Thus, we solve line by line for  $\beta$  and we see that (14) is a flow problem whose solution is of the form:

$$\beta_{i\bullet} = \beta'_{i\bullet} + \beta''_{i\bullet} \tag{15}$$

with  $(\beta''_{i\bullet})^t \in \ker(G^{H,i})^t$  and  $\beta'_{i\bullet}$  a particular solution.

More precisely, let  $\mathcal{T}^i$  be a spanning tree for  $\mathcal{S}^{H,i}$ ; call  $\Gamma^i$  the edge-node incidence matrix associated with  $\mathcal{T}^i$ ; we know that  $\Gamma^i$  has  $|\tilde{C}_i| - 1$  rows and  $|\tilde{C}_i|$  columns, and it is of rank  $|\tilde{C}_i| - 1$ . We choose a vertex  $m$  in  $\Gamma^i$  and we solve the system:

$$\sum_{e \in \mathcal{E}(\mathcal{T}^i)} \beta'_{ie} \Gamma_{en}^i = \Theta_{i,n}, \quad \forall n \in \tilde{C}_i \setminus \{m\}, \tag{16}$$

where  $\mathcal{E}(\mathcal{T}^i)$  denotes the set of indices of the edges of  $\mathcal{T}^i$ . The system (16) is a regular system of  $|\tilde{C}_i| - 1$  equations with  $|\tilde{C}_i| - 1$  unknowns, and we put  $\beta'_{ie}$  equal to 0 if  $e$  is in  $\tilde{I}_i \setminus \mathcal{E}(\mathcal{T}^i)$ .

It remains to show that the forgotten equation of index  $m$  in (16) is automatically satisfied. Indeed, by denoting  $i$  by  $\overline{pq}^h$ , we sum the right-hand side of (14) with respect to  $n \in \tilde{C}_i$ :

$$\sum_{n \in \tilde{C}_i} \Theta_{i,n} = \sum_{n \in \tilde{L}_p \cup \tilde{L}_q} \alpha_{qn} - \alpha_{pn} = 0, \tag{17}$$

since in view of (6) and (7),  $\sum_{n \in \tilde{L}_p \cup \tilde{L}_q} \alpha_{pn} = \sum_{n \in \tilde{L}_p} \alpha_{pn} = 1$  and  $\sum_{n \in \tilde{L}_p \cup \tilde{L}_q} \alpha_{qn} = \sum_{n \in \tilde{L}_q} \alpha_{qn} = 1$ .

On the other hand, if we sum the left-hand side of (14) with respect to  $n \in \tilde{C}_i$ , we obtain:

$$\sum_{n \in \tilde{C}_i} \sum_{e \in \tilde{I}_i} \beta_{ie} G_{en}^{H,i} = \sum_{e \in \tilde{I}_i} \beta_{ie} \sum_{n \in \tilde{C}_i} G_{en}^{H,i} = 0, \tag{18}$$

since each line of  $G^{H,i}$  contains only two non-zero coefficients  $+1$  and  $-1$ .

If  $i \notin \tilde{F}$ ,  $|\tilde{C}_i| = |\tilde{L}_p| = |\tilde{L}_q| = 1$  and the relation  $\sum_{e=1}^{E^H} \beta_{ie} G_{en}^H = \Theta_{i,n}$  is satisfied from (12) and (17).

Now we assume that  $\mathcal{S}^{H,i}$  is not connected and we denote by  $\hat{C}_i$  the nodes of a connected component. For the same reasons as in (18), if  $\beta$  satisfies (11) one gets  $\sum_{n \in \hat{C}_i} \sum_{e=1}^{E^H} \beta_{ie} G_{en}^H = 0$ .

However we can construct a matrix  $\alpha$  satisfying (6) and (7) such that  $\sum_{n \in \hat{C}_i} \sum_{r=1}^{N^h} G_{ir}^h \alpha_{rn} \neq 0$ . In fact, in view of (7), for  $i = \overline{pq}^h$  we can write:

$$\sum_{n \in \hat{C}_i} \sum_{r=1}^{N^h} G_{ir}^h \alpha_{rn} = \sum_{n \in \hat{C}_i} \alpha_{qn} - \alpha_{pn} = \sum_{\hat{C}_i \cap \tilde{L}_q} \alpha_{qn} - \sum_{\hat{C}_i \cap \tilde{L}_p} \alpha_{pn}. \tag{19}$$

Since  $\hat{C}_i$  is strictly included in  $\tilde{L}_p \cup \tilde{L}_q$ , we will have  $\hat{C}_i \cap \tilde{L}_p \neq \tilde{L}_p$  or  $\hat{C}_i \cap \tilde{L}_q \neq \tilde{L}_q$ . Depending on the situation, we can construct a suitable matrix  $\alpha$  such that:

$$\left( \sum_{\hat{C}_i \cap \tilde{L}_q} \alpha_{qn} = 1 \text{ and } \sum_{\hat{C}_i \cap \tilde{L}_p} \alpha_{pn} = 0 \right) \quad \text{or} \quad \left( \sum_{\hat{C}_i \cap \tilde{L}_q} \alpha_{qn} = 0 \text{ and } \sum_{\hat{C}_i \cap \tilde{L}_p} \alpha_{pn} = 1 \right).$$

For these matrices  $\alpha$ , the condition defined by (4) cannot be ensured.

#### 4. Construction of the coarse edge functions

For a coarse graph satisfying the condition of Proposition 2.3 and by using the decomposition (15), any compatible matrix can be written  $\beta = \beta' + \beta''$ , where the complete matrices are defined by gathering the lines of index  $i$   $\beta_{i\bullet}$ ,  $\beta'_{i\bullet}$  and  $\beta''_{i\bullet}$ . The computation of each  $\beta'_{i\bullet}$  can be done by solving system (16). As concerns  $\beta''_{i\bullet}$ , a basis of the kernel of  $(G^{H,i})^t$  is given by a set of  $k_i$  independent cycles of  $S^{H,i}$ . Then,  $\sum_{i \in \tilde{F}} k_i$  degrees of freedom should be determined by minimizing an appropriate energy functional; such a problem is introduced in [4] and can be related to explanations in [3].

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#### References

- [1] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numer.* 11 (2002) 237–339.
- [2] R. Hiptmair, Multigrid method for Maxwell's equations, *SIAM J. Numer. Anal.* 36 (1999) 204–225.
- [3] J. Mandel, M. Brezina, P. Vaněk, Energy optimization of algebraic multigrid bases, *Computing* 62 (3) (1999) 205–228.
- [4] F. Musy, L. Nicolas, R. Perrussel, M. Schatzman, Compatible coarse nodal and edge elements through energy functionals, UMR MAPLY, internal report 394, 2004, <http://maply.univ-lyon1.fr/~perrussel/report.pdf>.
- [5] S. Reitzinger, J. Schöberl, An algebraic multigrid method for finite element discretizations with edge elements, *Numer. Linear Algebra Appl.* 9 (2002) 223–238.