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Partial Differential Equations

The Helmholtz equation with impedance in a half-space

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Abstract

In this Note we obtain existence and uniqueness results for the Helmholtz equation in the half-space $\mathbb{R}^3_+$ with an impedance or Robin boundary condition. Basically, we follow the procedure we have already used in the bi-dimensional case (the half-plane). Thus, we compute the associated Green's function with the help of a double Fourier transform and we analyze its far field in order to obtain radiation conditions that allow us to prove the uniqueness of an outgoing solution. Again, these radiation conditions are somewhat unusual due to the appearance of a surface wave guided by the boundary. An integral representation of the solution is presented by means of the Green's function and the boundary data. ***To cite this article: M. Durán et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

L'équation de Helmholtz avec impédance dans un demi-espace. Dans cette Note, nous démontrons un résultat d'existence et d'unicité de la solution de l'équation de Helmholtz dans un demi-espace avec une condition d'impédance. Le domaine est non borné et sa frontière également. Les conditions de radiation sont différentes des conditions usuelles pour un problème extérieur, ceci étant lié à la présence d'ondes de surface. Nous calculons la fonction de Green du demi-espace et nous étudions son comportement à l'infini. Ceci conduit à l'expression des conditions de radiation qui permettent de démontrer l'unicité. L'utilisation de la représentation intégrale donne le résultat d'existence. ***Pour citer cet article : M. Durán et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

L'origine de ce problème est l'acoustique. On considère un demi-espace limité par une paroi plane et infinie. La pression acoustique satisfait l'équation de Helmholtz. Dans le cas d'une paroi passive [4], la solution satisfait une condition limite du type Robin avec un paramètre d'impédance z. Lorsque cette impédance est réelle, il apparait des ondes de surface guidées par la paroi et exponentiellement décroissantes dans l'autre direction. La vitesse de ces ondes est différente de celles des ondes planes usuelles dans l'espace libre. La partie rerayonnée de la solution ne peut donc pas satisfaire aux conditions de Sommerfeld usuelles. Nous calculons la fonction de Green du demi-espace et nous étudions son comportement à l'infini. Ceci nous permet de trouver l'expression des conditions de radiation,

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qui permettent ensuite de reformuler le problème et de démontrer l'unicité dans le cas du demi-espace. Nous utilisons pour ceci les fonctions de Legendre et les fonctions de Bessel sphériques. L'utilisation de la représentation intégrale donne le résultat d'existence.

Dans une seconde partie, nous considérons le même problème, mais dans un domaine où la paroi infinie est localement perturbée. On étend à ce cas le théorème d'existence et unicité des solutions sortantes en introduisant un couplage via l'étude de l'opérateur de Dirichlet–Neumann dans le cas de l'extérieur d'une demi-sphère.

Les articles référencés [3–5] traitent du cas du demi-espace, mais sans donner une expression claire des conditions de radiation ce qui ne permet pas de traiter le cas perturbé.

La version bidimensionelle de ce travail est exposée dans [1].

1. Introduction

We consider a half-space $\mathbb{R}^3_+ = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^2\colon\ x_3 > 0\}$. Our problem is to find in this half-space, a solution for the Helmholtz equation with Robin or mixed Dirichlet–Neumann boundary data, sometimes called impedance boundary value problem:

$$\begin{cases} \Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0 & \text{in } \mathbb{R}^3_+, \\ \dfrac{\partial u}{\partial \mathbf{n}} - zu = f & \text{over } \{x_3 = 0\}. \end{cases} \tag{1}$$

The normal vector $\mathbf{n}$ is outwardly directed, so in this problem $\frac{\partial u}{\partial \mathbf{n}}$ becomes $-\frac{\partial u}{\partial x_3}$. The given wave number k is positive. The impedance term z will be a positive real number that must be looked, in absence of any source f, as a proportionality constant between the wave and its normal derivative. For our intentions, the function f must have compact support in $\mathbb{R}^2$.

Problem (1) is not well posed yet. In order to have uniqueness of an outgoing solution, we need to add radiation conditions. These conditions will be taken from an asymptotic study of the associated Green's function. We find some related works in [3–5]. However, they lack the asymptotic analysis and therefore they have no clear expressions for the radiation conditions.

2. The associated Green's function

We take a source point $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_+$ and since there is no horizontal variation in the geometry of the problem, we can suppose for the moment that $y_1 = y_2 = 0$. Our Green function will be a distribution satisfying:

$$\begin{cases} \Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) + k^2 G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}) & \text{in } \mathbb{R}^3_+, \\ \dfrac{\partial G}{\partial x_3} + zG = 0 & \text{over } \{x_3 = 0\}. \end{cases} \tag{2}$$

We choose a complex value of k, i.e. $k = k_0 + \mathrm{i}\varepsilon$, $\varepsilon > 0$. The radiation condition will be obtained studying the far field of the Green's function, which itself is the limit of the above one when ε tends to zero. This idea is related with the limit absorption principle [7].

Taking a double Fourier transform in the horizontal directions, we get the following differential equation (with initial condition) in the vertical variable x_3:

$$\begin{cases} \dfrac{\partial^2 \widehat{G}}{\partial x_3^2} + \left(k^2 - \xi_1^2 - \xi_2^2\right)\widehat{G} = \dfrac{1}{2\pi}\delta_{y_3} & \text{for } x_3 > 0, \\ \dfrac{\partial \widehat{G}}{\partial x_3} + z\widehat{G} = 0 & \text{at } x_3 = 0. \end{cases} \tag{3}$$

Hereafter we will use some special kind of polar coordinates:

$$\xi = \begin{cases} \sqrt{\xi_1^2 + \xi_2^2} & \text{if } \xi_2 > 0, \\ \xi_1 & \text{if } \xi_2 = 0, \\ -\sqrt{\xi_1^2 + \xi_2^2} & \text{if } \xi_2 < 0, \end{cases} \quad \text{and} \quad \psi = \cot g^{-1}\left(\frac{\xi_1}{\xi_2}\right). \tag{4}$$

The (ξ_1, ξ_2)-plane is described now as $\xi \in\ ]-\infty, +\infty[$ and $\psi \in [0, \pi[$.

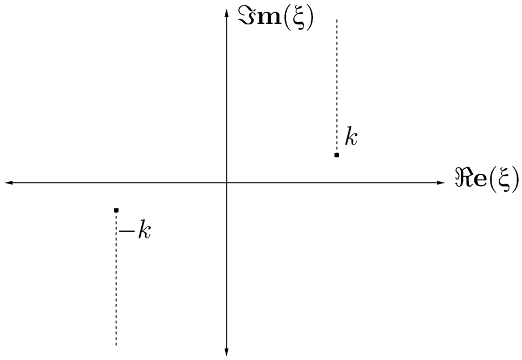

Fig. 1. $D_1 \cap D_2$: Domain of the complex map (5).

2.1. Determination of the square root

To be precise, we have to give an exact meaning to the complex map

$$\xi \mapsto \sqrt{\xi^2 - k^2}. \tag{5}$$

As we have made in [1], we define $\sqrt{\xi^2 - k^2}$ as the product between $\sqrt{\xi - k}$ and $\sqrt{\xi + k}$. The first square root is defined using an analytic branch of the logarithm in a region D composed of the whole complex plane minus the non-negative imaginary axis. In that way, $\sqrt{\xi - k}$ is taken as a complex square root which is analytic in the region $D_1 = D + k$. On the other hand, $\sqrt{\xi + k}$ is defined using an analytic branch of the logarithm in the region D' composed of the whole complex plane minus the non-positive imaginary axis. Then, $\sqrt{\xi + k}$ is analytically defined in the region $D_2 = D' - k$. Thus, our complex function (5) is even and analytic in the intersection $D_1 \cap D_2$ (see Fig. 1). It has the expression

$$\sqrt{\xi^2 - k^2} = -\mathrm{i}k \exp\left(\int_0^{\xi} \frac{\eta}{\eta^2 - k^2} \, \mathrm{d}\eta \right). \tag{6}$$

Remark 1. For real ξ, the real part of the complex map (5) is strictly positive, which implies that the function $\mathrm{e}^{-\sqrt{\xi^2 - k^2} x_3}$ is even and exponentially decreasing when $x_3 \to +\infty$.

2.2. The integral expression for G

The solution of (3) is called the spectral Green's function and its analytical expression is

$$\widehat{G}(\xi, x_3, y_3) = \frac{1}{4\pi} \left(\frac{z + \sqrt{\xi^2 - k^2}}{z - \sqrt{\xi^2 - k^2}} \frac{\mathrm{e}^{-\sqrt{\xi^2 - k^2}(x_3 + y_3)}}{\sqrt{\xi^2 - k^2}} - \frac{\mathrm{e}^{-\sqrt{\xi^2 - k^2}|x_3 - y_3|}}{\sqrt{\xi^2 - k^2}} \right). \tag{7}$$

So when the source point is not necessary in $y_1 = y_2 = 0$, the spatial Green's function is represented in terms of the inverse Fourier transform integral:

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{G}(\xi, x_3, y_3) \, \mathrm{e}^{-\mathrm{i}[\xi_1(x_1 - y_1) + \xi_2(x_2 - y_2)]} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2. \tag{8}$$

3. The radiation conditions

We introduce spherical coordinates in the spatial variables (see Fig. 2):

$$\begin{cases} x_1 - y_1 = r \sin\theta \cos\varphi, \\ x_2 - y_2 = r \sin\theta \sin\varphi, \\ x_3 - y_3 = r \cos\theta. \end{cases} \tag{9}$$

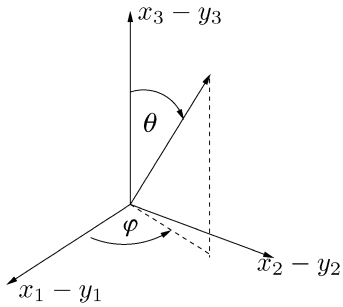

Fig. 2. Spherical coordinates.

In order to find the radiation conditions for the problem (1), we want to know how the integral which expression is (8), behaves when $r \to +\infty$. Due to the absolute value, the integrand function have different behaviors depending on $x_3 - y_3 > 0$ or $x_3 - y_3 \leqslant 0$. Moreover, throughout this section we take the limit case $\varepsilon \to 0^+$, then the square root appearing in the exponentials is purely imaginary for $|\xi| < k$ and purely real for $|\xi| \geqslant k$. That suggests splitting the analysis of expression (8) in two integrals. The integral running over the circle of radius k will give information of the 'classical' wave behavior. Its asymptotic mean term will be

$$\left(\frac{z - \mathrm{i}k\cos\theta}{z + \mathrm{i}k\cos\theta}\,\mathrm{e}^{2\mathrm{i}k\cos\theta y_3} - 1\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}. \tag{10}$$

The integral over the exterior of the circle of radius k will give information of the evanescent waves. Its mean contribution will be the surface wave:

$$\frac{z\mathrm{e}^{-z(x_3+y_3)}}{2\pi\mathrm{i}}\int_0^{\pi}\mathrm{e}^{\mathrm{i}\sqrt{z^2+k^2}\,r\sin\theta|\cos(\psi-\varphi)|}\,\mathrm{d}\psi. \tag{11}$$

Both terms, (10) and (11), correspond to two different outgoing waves. The first one decays in amplitude when approaching the plane $x_3 - y_3 = r\cos\theta = 0$. The second one is guided by the plane and its amplitude is exponentially decreasing in the vertical direction. This phenomenon induces to consider two different radiation conditions: one to be applied in a domain like $r\cos\theta > r^\alpha$ $(0 < \alpha < 1)$; the other one to be applied in a complementary domain. Nevertheless, it is necessary to exhibit which is the asymptotic behavior of the residual terms. It will be useful to work also with the horizontal radial variable ρ, defined as the radius of the projection over the horizontal plane, i.e.

$$\rho := r\sin\theta = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}. \tag{12}$$

Using some analysis techniques like the stationary phase [2], integration by parts and the calculus of residues, we get the following estimations for $0 < \alpha < 1$:

$$\begin{cases} G(\mathbf{x},\mathbf{y}) = \left(\dfrac{z - \mathrm{i}k\cos\theta}{z + \mathrm{i}k\cos\theta}\,\mathrm{e}^{2\mathrm{i}k\cos\theta y_3} - 1\right)\dfrac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} + \mathrm{O}\big(r^{-(2\alpha+1/2)}\big), & \text{when } r\cos\theta > r^\alpha \text{ and } r \to +\infty, \\ G(\mathbf{x},\mathbf{y}) = \dfrac{z\mathrm{e}^{-z(x_3+y_3)}}{\mathrm{i}\sqrt{2\pi}(z^2+k^2)^{1/4}}\,\dfrac{\mathrm{e}^{\mathrm{i}(\sqrt{z^2+k^2}\rho-\pi/4)}}{\sqrt{\rho}} + \mathrm{O}\big(\rho^{-3/2}\big), & \text{when } r\cos\theta < r^\alpha \text{ and } \rho \to +\infty. \end{cases} \tag{13}$$

So we write over the following general radiation condition for r large and $1/4 < \alpha < 1/2$:

$$\begin{cases} \left|\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right| < cr^{-(2\alpha+1/2)} & \text{in } \mathbb{R}^3_+(\alpha+) := \{x_3 > cr^\alpha\} \quad \text{and} \\ \left|\dfrac{\partial u}{\partial r} - \mathrm{i}\sqrt{k^2+z^2}\,u\right| < cr^{\alpha-3/2} & \text{in } \mathbb{R}^3_+(\alpha-) := \{0 \leqslant x_3 < cr^\alpha\}. \end{cases} \tag{14}$$

These conditions are sufficient to avoid the case of ingoing wave behavior.

Remark 2. Denote by S_R^+ the surface of the half-sphere of radius R contained in the half-space $\mathbb{R}^3_+$. Let S_R^1 be the part of S_R^+, contained in the domain $\mathbb{R}^3_+(\alpha+)$. Let S_R^2 be the complementary part. A weaker version of radiation conditions (14) is

$$\lim_{R\to+\infty}\int\limits_{S_R^1}\left|\frac{\partial u}{\partial r}-\mathrm{i}ku\right|^2\mathrm{d}S=0\quad\text{and}\quad\lim_{R\to+\infty}\int\limits_{S_R^2}\left|\frac{\partial u}{\partial r}-\mathrm{i}\sqrt{z^2+k^2}\,u\right|^2\mathrm{d}S=0.\tag{15}$$

4. Functional spaces

Since our domains are unbounded, we need to work with weighted functional spaces. We will use powers of the classic weight functions

$$\varrho=\sqrt{1+r^2}\quad\text{and}\quad\log\varrho=\log\bigl(2+r^2\bigr).\tag{16}$$

A good description of the space where we will look for a solution must consider all the asymptotic behaviors analyzed in the previous section. As we have seen, the worse behavior in terms of integrability at infinity, was the one given by the surface wave (11). Nevertheless this bad behavior occurs only in the horizontal direction, because in the vertical direction we have exponential decay. To describe this phenomenon we fix the value of $x_3\geqslant 0$ and we define $u_{x_3}:\mathbb{R}^2\to\mathbb{C}$ as

$$u_{x_3}(x_1,x_2):=u(x_1,x_2,x_3).\tag{17}$$

With the asymptotic expression of the surface wave in mind (Eq. (13)), we write down the condition

$$\frac{u_{x_3}}{\sqrt{\varrho}\log\varrho}\in L^2\bigl(\mathbb{R}^2\bigr),\quad\text{for all } x_3\geqslant 0.\tag{18}$$

Now, given $\gamma>1/2$, a functional space which describes the behavior that we want for a solution of problem (1) is the following

$$W_\gamma\bigl(\mathbb{R}^3_+\bigr):=\left\{u\colon\ \frac{u}{\varrho^\gamma}\in L^2\bigl(\mathbb{R}^3_+\bigr),\ \frac{\nabla u}{\varrho^\gamma}\in L^2\bigl(\mathbb{R}^3_+\bigr),\ u\text{ satisfies conditions (15) and (18)}\right\}.\tag{19}$$

5. The uniqueness and existence results

The following functions will be essential in the uniqueness theorem. We start denoting by $\mathbb{P}_l$, the Legendre polynomials, which are the orthogonal polynomials defined on the segment $]-1,1[$ for the scalar product in $L^2(]-1,1[)$, constructed with the Gram–Schmidt orthogonalization process, when starting from the usual basis $1,x,x^2,\ldots$. The usual normalization consists in fixing $\mathbb{P}_l(1)=1$.

We denote by $\{j_l\}_{l\geqslant 0}$, the collection of spherical Bessel functions of the first kind. An useful expansion relating these functions with the Legendre polynomials is [6]:

$$\mathrm{e}^{\mathrm{i}kx}=\sum_{l=0}^{\infty}\mathrm{i}^l(2l+1)\,j_l(k)\,\mathbb{P}_l(x).\tag{20}$$

Finally, we introduce the associated Legendre functions $\mathbb{P}_l^m(\cos\theta)$ defined in terms of the Legendre polynomial by the recursion formulas

$$\begin{cases}\mathbb{P}_l^m(\cos\theta):=(\sin\theta)^m\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^m\mathbb{P}_l(\cos\theta)\quad\text{for }0\leqslant m\leqslant l,\quad\text{and}\\[2mm]\mathbb{P}_l^{-m}(\cos\theta)=(-1)^m\dfrac{(l-m)!}{(l+m)!}\mathbb{P}_l^m(\cos\theta)\quad\text{for }0\leqslant m\leqslant l.\end{cases}\tag{21}$$

Their parity is such that

$$\mathbb{P}_l^m(0)=0,\quad\text{when }(l+m)\text{ is an odd integer.}\tag{22}$$

Theorem 5.1 (Uniqueness). *The problem* (1) *admits a unique outgoing solution which is in* $W_\gamma(\mathbb{R}^3_+)$.

Proof. We consider the case where $f = 0$. The proof is divided in several steps. The aim will be to show that $u_0 \equiv 0$ which also implies that $\frac{\partial u}{\partial x_3}|_{x_3=0} \equiv 0$ by the boundary condition. The Fourier transform properties of a solution will give that in fact u vanishes everywhere in $\mathbb{R}^3_+$.

An important step in the proof, is an orthogonality property of u_0 with the functions

$$v_l^m(r,\theta,\varphi) = j_l(kr)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathbb{P}_l^m(\cos\theta), \quad l \geqslant 0,\ |m| < l.$$

That information will give that the Fourier transform $\hat{u}_0$ is null over the ball of radius k. The nullity of $\hat{u}_0$ over the complementary domain, can be obtained from the associated ordinary differential equation that the partial Fourier transform of u must satisfy. □

Theorem 5.2 (Existence). *Let* $d > 0$ *such that the support of* f *is contained in the open ball* $B_d \subset \mathbb{R}^2$ *of radius* d. *The function* u *defined by*

$$u(\mathbf{x}) = \int_{B_d} G(\mathbf{x};\ y_1, y_2, 0)\, f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2, \tag{23}$$

satisfies Eq. (1) *and the radiation condition* (14).

Remark 3. The previous results can be extended to domains which boundaries are local perturbations of a plane. A first step consist of extending the uniqueness result to the case of a domain which is the exterior (in $\mathbb{R}^3_+$) of the unitary half-sphere S^+, with a Dirichlet boundary data over the surface of S^+ and with the same impedance boundary conditions over $\mathbb{R}^2$ minus the unitary disc. Problems in perturbed geometries are then solved through a coupling technique using a Dirichlet to Neumann operator.

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