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Complex Analysis

On the compactness of the automorphism group of a domain

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Abstract

We give a sufficient condition on the boundary of a domain, insuring that the automorphism group of the domain is compact. ***To cite this article: J. Byun, H. Gaussier, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur la compacité du groupe d'automorhismes d'un domaine. Nous donnons une condition suffisante sur la frontière d'un domaine assurant la compacité du groupe de Lie des automorphismes holomorphes du domaine. ***Pour citer cet article : J. Byun, H. Gaussier, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

The classification of domains with noncompact holomorphic automorphism group has been intensively studied since the pioneer works of Henri Poincaré. A natural approach consists in studying the trajectories of automorphism orbits. Generically the holomorphic automorphism group of a domain is compact. This is equivalent to the compactness of the trajectories. In contrast there are few classes of relatively compact domains with noncompact automorphism group. For such domains some orbits accumulate at boundary points called *orbit accumulation points*. The classification relies deeply on the study of the geometry of the boundary at an orbit accumulation point p. For instance, by [16,15], if p is a strongly pseudoconvex point, then the domain is biholomorphically equivalent to the unit ball. In [1], Bedford and Pinchuk proved that if Ω is bounded, pseudoconvex with real-analytic boundary in $\mathbb{C}^2$ and if Ω has a noncompact automorphism group, then Ω is biholomorphic to the ellipsoid $\{(z,w) \in \mathbb{C}^2\colon\ |z|^{2m} + |w|^2 < 1\}$. We recall that a domain is of finite D'Angelo type (see [6]) at a boundary point if the order of contact at that point between the boundary and complex curves is upper bounded, uniformly with respect to the curves. Greene and Krantz [8] suggested the following problem to understand the geometry of a domain in a complex space.

Conjecture 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and let p be a boundary point. If p is an orbit accumulation point, then Ω is of finite D'Angelo type at p.*

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The main results around this conjecture are due to Greene and Krantz [8], Kim [10], Kang [9], Kim and Krantz [12], Landucci [14].

The aim of this Note is to give sufficient conditions on the geometry of a domain at a boundary point to insure that there is no orbit accumulating at that point. Let M be a hypersurface in $\mathbb{C}^n$ and let $P_m(M)$ be the set of all points in M of type m (m is either a positive integer or infinity). In [14], Landucci proved that the automorphism group of a domain is compact if $P_\infty(\partial\Omega)$ is a closed interval on the real 'normal' line in a complex space with dimension 2. Our theorem shows that if the closed interval with constant type is isolated (statement (b) in Theorem 1.1 below), then there is no automorphism orbit accumulating at the convex part of the boundary.

Theorem 1.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, satisfying Condition* (R). *Assume that there are two open sets U, V such that*

(a) $\overline{U} \subset V$,
(b) *A connected component of $P_m(\partial\Omega \cap V)$ is an interval $I \subset U$ transversal to the complex tangent space $T_z^{\mathbb{C}}(\partial\Omega)$ at one point $z \in I$.*

Then there are no automorphism orbits in Ω accumulating at a boundary point p which has a neighborhood W satisfying $W \cap \Omega$ is convex.

The precise definition of Condition (R) is given in [4]. If Ω satisfies Condition (R) then every automorphism $f \in \mathrm{Aut}(\Omega)$ extends smoothly up to the boundary i.e. $f \in C^\infty(\overline{\Omega})$. As a direct consequence of the proof of Theorem 1.1 we give the following corollaries.

Corollary 1.2. *Under the hypothesis of Theorem* 1.1, *if Ω is convex, then the automorphism group of the domain is compact.*

In case condition (b) deals with the global set $P_m(\partial\Omega)$, we obtain:

Corollary 1.3. *Let Ω be a smooth bounded convex domain in $\mathbb{C}^n$. Assume that a connected component of $P_m(\partial\Omega)$ is an interval I transversal to $T_z^{\mathbb{C}}(\partial\Omega)$ at one point $z \in I$. Then* $\mathrm{Aut}(\Omega)$ *is compact.*

2. Proof of Theorem 1.1

From now on, let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary satisfying Condition (R) and let $z = (0, 0) \in \partial\Omega$. Without loss of generality, we can assume that there are two neighborhoods U, V of $z \in \partial\Omega$ such that $\overline{U} \subset V$.

Moreover, in virtue of condition (b), there exist complex coordinates (z', z_n) for $\mathbb{C}^n$ such that $T_z^{\mathbb{C}}(\partial\Omega) = \{z_n = 0\}$ and I is contained in the set $\{(0, z_n) \mid \operatorname{Im} z_n = 0\}$. Therefore, the axis $\operatorname{Im} z_n$ is normal to $\partial\Omega$ at z and $I = \{(0, z_n) \in U \cap \partial\Omega \mid \operatorname{Im} z_n = 0,\ a \leqslant \operatorname{Re} z_n \leqslant b\}$, for some real numbers a, b.

Expecting a contradiction, we assume that there is an automorphism orbit accumulating at a boundary point p in $W \cap \partial\Omega$. Since $\Omega \cap W$ is convex, it follows from a result of K.T. Kim [11] that there is a noncompact subgroup $T = \{f_t \in \mathrm{Aut}(\Omega) \mid t \in \mathbb{R}\}$ of $\mathrm{Aut}(\Omega)$ which is isomorphic to $\mathbb{R}$. By Condition (R), the restriction to $\partial\Omega$ of the extension of T to $\overline{\Omega}$ defines a CR automorphism of $\partial\Omega$. The map $t \mapsto f_t(I)$ is continuous, $f_0 = \mathrm{Id}$ and I is connected. Since the D'Angelo type is a CR invariant, we have $f_t(I) = I$ for all $t \in \mathbb{R}$, by condition (b).

Let C_n be the non-empty component of $\{(z', z_n) \in \Omega \mid z' = 0\}$, whose closure contains the origin.

Then we have the following

Lemma 2.1. *If $f \in \mathrm{Aut}(\Omega)$ with $f(I) = I$, then $f(C_n) = C_n$.*

Proof. For $1 \leqslant j \leqslant n-1$, consider the holomorphic function ψ_j defined in a domain $W = \{\zeta \in \mathbb{C}\colon\ (0, \zeta) \in C_n\}$ by $\psi_j(\zeta) := f_j(0, \zeta)$, where f_j is a jth component of f. Since $f(I) = I$ and $I = \{(z', z_n) \in U \mid z' = 0,\ \operatorname{Im} z_n = 0\}$, we easily see that $\psi_j(r) = 0$ for $r \in \mathbb{R}$. By the Schwarz Reflection Principle, ψ_j is identically zero. This completes the proof of Lemma 2.1. □

According to Lemma 2.1, $f_t(C_n) = C_n$. Hence we can view $T|_{C_n}$ as a noncompact subgroup of $\mathrm{Aut}(C_n)$ and C_n is a one-dimensional domain with a C^1 orbit accumulation point. By [13], there exists a biholomorphism φ between $\overline{C_n}$ and the closed unit disc $\bar{\Delta}$ in $\mathbb{C}$.

Consider the automorphism φ_t of Δ, defined by $\varphi_t := \varphi \circ f_t \circ \varphi^{-1}$. We may assume that $\varphi(I) = \{e^{i\theta}\colon \theta \in [-\pi, 0]\}$. Let X be the vector field generating $(\varphi_t)_{t\in\mathbb{R}}$. This extends up to $\bar{\Delta}$. By construction $\varphi_t(\varphi(I)) = \varphi(I)$, implying that $X(-1) = X(1) = 0$. Hence $\varphi_t(-1) = -1$ and $\varphi_t(1) = 1$ for every real t. If we write $\varphi_t(\zeta) = e^{i\theta_t}(\zeta - a_t)/(1 - \overline{a_t}\zeta)$ then the two preceding equalities imply that a_t is real for every t and $e^{i\theta_t} = 1$. In particular, we have: $\lim_{t\to+\infty} \varphi_t(0) = 1$ and $\lim_{t\to-\infty} \varphi_t(0) = -1$ non tangentially. So we can assume that $\lim_{t\to+\infty} f_t = (0, a)$ and $\lim_{t\to-\infty} f_t = (0, b)$ uniformly on compact subsets of Ω.

We divide the study into the following two cases:

(I) $m = \infty$. By scaling theory [12], there is a biholomorphism between Ω and a model domain $\Omega_\rho := \{\mathrm{Re}\, z_n + \rho(z'') < 0\}$, where $z'' = (z_1, \ldots, z_{n-2})$ and ρ is a real valued function on $\mathbb{C}^{n-2}$. Note that the model domain Ω_ρ contains a complex line, since the defining function does not depend on the variable z_{n-1}. Therefore the domain Ω is not Kobayashi hyperbolic. This is a contradiction.

(II) $m < \infty$. The following theorem is due to S. Bell [3]. We reformulate this for our purpose.

Theorem 2.2. *Let D be a bounded domain with a C^∞ smooth boundary, satisfying Condition* (R)*. Assume that there is a sequence f_j of automorphisms and two boundary points p_1 and p_2 such that f_j, f_j^{-1} converge uniformly on compact subsets of D to the constant functions p_1, p_2, respectively. If Ω is of finite type at p_1 and p_2, then f_j converges uniformly on compact subsets of $\overline{D} \setminus \{p_2\}$ to the constant function p_1.*

We apply Theorem 2.2 to $D = \Omega$, $p_1 = (0, a)$ and $p_2 = (0, b)$. It follows from the invariance of the type that Ω is a weakly pseudoconvex domain of finite type. Note that Ω is locally convex at p. By dilation of coordinates [2], we can construct a biholomorphism Φ from Ω to $D = \{z \in \mathbb{C}^n\colon \mathrm{Re}\, z_n + \rho(z') < 0\}$, where ρ is a smooth real valued function on $\mathbb{C}^{n-1}$. Using some estimates of the Kobayashi–Royden infinitesimal pseudometric at infinity in a convex domain [2] and at points in the finite type pseudoconvex boundary $\partial\Omega$ [5], we obtain the existence of a parabolic orbit. Namely, there is a point $p' \in \partial\Omega$ such that $\lim_{t\to\pm\infty} \Phi^{-1} \circ l_t \circ \Phi(q) = p'$, where q is an interior point of Ω and $l_t(z', z_n) = (z', z_n + it)$ (see [1]).

Let $g_t := \Phi^{-1} \circ l_t \circ \Phi$. Note that $g_t(I) = I$ because g_t is a subgroup of the automorphism group. Repeating the previous argument, we obtain that, like φ_t, g_t has at least two fixed points in I. Hence, g_t is not parabolic which is a contradiction. □

3. Examples

We present several useful examples for a general approach to the Greene–Krantz conjecture.

From now on, let (z, w) be a standard coordinate for $\mathbb{C}^2$ and let Ψ be the Cayley transform defined by $\Psi(z, w) = (z/(1 - w), (1 + w)/(1 - w))$.

Example 1. Let $\Omega_m = \{(z, w) \in \mathbb{C}^2 \mid 4\,\mathrm{Re}\, w + |z|^2 + |z|^{2m}\}$. Then the automorphism group of $\Psi(\Omega_m)$ is noncompact and $\Psi(\Omega_m)$ admits the following defining function $\rho_m(z, w) = |1 + w|^{2m-2}(|w|^2 + |z|^2) + 2^{2m-2}|z|^{2m} - 1$. The boundary of $\Psi(\Omega_m)$ is C^∞ smooth except at $p = (0, -1)$ where it is of class $C^{1,1-1/m}$. The point p is an orbit accumulation point of the domain.

Example 2. Let Ψ_m be a generalized Cayley transform defined by $\Psi_m(z, w) = (z/\sqrt[m]{1 - w}, (1 + w)/(1 - w))$. Then $\Psi_m(\Omega_m)$ has a noncompact automorphism group. By a straightforward computation, the domain $\Psi_m(\Omega_m)$ admits the following defining function:

$$\gamma_m(z, w) = |w|^2 + \left|(1 + w)/2\right|^{2-2/m}|z|^2 + |z|^{2m} - 1.$$

The boundary of domain is $C^{1,1-1/m}$ smooth at $(0, -1)$.

Let $\Omega = \lim_{m\to\infty} \Psi_m(\Omega_m)$. Then the domain Ω has a noncompact automorphism group by [7] and the domain Ω is defined by $\Omega = \{(z, w) \in \mathbb{C}^2\colon\ |w|^2 + \frac{1}{4}|1 + w|^2|z|^2 - 1 < 0,\ |z| < 1\}$.

Let $\Phi(z,w) = (1/2(1+w)z, w)$. Then the domain $\Phi(\Omega)$ is biholomorphic neither to the ball nor to the polydisc.

The next two examples are constructed as follows. Let $\Omega_\rho = \{4\,\mathrm{Re}\,w + \rho(|z|) < 0\}$. We consider a hypersurface $H = \Omega_\rho \cap \{\mathrm{Im}\,w = 0\}$. Notice that the domain $\Omega_\rho = \bigcup_{t\in\mathbb{R}} l_t(H)$ where $l_t(z,w) = (z, w + \mathrm{i}t),\ t \in \mathbb{R}$. Then $\Psi(\Omega_\rho) = \bigcup_{t\in\mathbb{R}} \Psi \circ l_t \circ \Psi^{-1}(\Psi(H))$. By a careful choice of $\Psi(H)$, we construct the two following examples.

Example 3. Since $\Psi(H)$ is contained in $\{(x+\mathrm{i}y, u+\mathrm{i}v):\ v = 0\}$, we define the boundary for $\Psi(H)$ by the following equation with respect to u, x, y.

$$|z| = \begin{cases} \sqrt{4-u^2} & \text{for } -2 \leqslant u \leqslant 0, \\ \chi(u) & \text{for } 0 \leqslant u \leqslant 1, \end{cases}$$

where χ is a smooth function satisfying $\chi(1) = 1$, $\chi(0) = 2$ and $\lim_{t\to 1} \chi'(t) = \infty$.

Then the domain $\bigcup_{t\in\mathbb{R}} f_t(\Psi(H))$, where $f_t = \Psi \circ l_t \circ \Psi^{-1}$, has a noncompact automorphism group. Moreover its boundary is given by $E_1 = \bigcup_{t\in\mathbb{R}} f_t(\partial\Psi(H))$ and $E_2 = \{(\zeta, 1) \in \mathbb{C}^2:\ |\zeta| \leqslant 1\}$. The boundary is smooth on E_1 but only globally continuous.

Example 4. Let the boundary of $\Psi(H)$ be defined by $u = 1 - \exp(-1/|z|^2)$ on a neighborhood of $(0,1)$ and assume that the whole boundary is a smooth surface with respect to $u,\ x,\ y$. Then the domain $\bigcup_{t\in\mathbb{R}} f_t(\Psi(H))$ has a noncompact automorphism group. The boundary of the domain is C^1 smooth at $(0,1)$.

References

[1] E. Bedford, S. Pinchuk, Domains in $\mathbb{C}^2$ with non compact holomorphic automorphism group, Math. USSR-Sb. 63 (1989) 141–151.
[2] E. Bedford, S. Pinchuk, Convex domains with noncompact groups of automorphisms, Mat. Sb. 185 (1994) 3–26 (in Russian); Translation in Russian, Acad. Sci. Sb. Math. 82 (1995) 1–20.
[3] S. Bell, Compactness of families of holomorphic mappings up to the boundary, in: Lecture Notes in Math., vol. 1268, Springer-Verlag, Berlin/New York, 1987, pp. 29–42.
[4] S.R. Bell, E. Ligocka, A simplification and extension of Fefferman's theorem on biholomorphic mappings, Invent. Math. 57 (1980) 283–289.
[5] S. Cho, A lower bound on the Kobayashi metric near a point of finite type in C^n, J. Geom. Anal. 2 (4) (1992) 317–325.
[6] J.P. D'Angelo, Real hypersurfaces, orders of contact, and applications, Ann. Math. 115 (1982) 615–673.
[7] B.L. Fridman, D. Ma, E.A. Poletsky, Upper semicontinuity of the dimensions of automorphism groups of domains in $\mathbb{C}^N$, Amer. J. Math. 125 (2) (2003) 289–299.
[8] R.E. Greene, S.G. Krantz, Techniques for studying automorphisms of weakly pseudoconvex domains, in: Math. Notes, vol. 38, Princeton Univ. Press, Princeton, NJ, 1993, pp. 389–410.
[9] H.B. Kang, Holomorphic automorphisms of certain class of domains of infinite type, Tohoku Math. J. 46 (1994) 435–442.
[10] K.T. Kim, On a boundary point repelling automorphism orbits, J. Math. Anal. Appl. 179 (1993) 463–482.
[11] K.T. Kim, On the automorphism group of convex domain in $\mathbb{C}^n$, Adv. in Geom. 4 (2004) 33–40.
[12] K.T. Kim, S. Krantz, Complex scaling and Domains with non-compact automorphism group, Illinois J. Math. 45 (2001) 1273–1299.
[13] S.G. Krantz, Characterization of smooth domains in C by their biholomorphic self-maps, Amer. Math. Monthly 90 (8) (1983) 555–557.
[14] M. Landucci, The automorphism group of domains with boundary points of infinite type, Illinois J. Math. 48 (2004) 875–885.
[15] J.P. Rosay, Une caractérisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes, Ann. Inst. Fourier (Grenoble) 29 (1979) 91–97.
[16] B. Wong, Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group, Invent. Math. 41 (1977) 253–257.