Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 341 (2005) 405–408
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Complex Analysis

Tchebotaröv's problem

Promarz Tamrazov

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska Str. 3, 01601 Kiev, Ukraine

Received 25 November 2004; accepted after revision 26 August 2005

Presented by Pierre Lelong

Abstract

We give a complete solution to the extremal problem posed by N.G. Tchebotaröv in the mid 1920s, and we establish explicit parametric formulae for the extremals. ***To cite this article: P. Tamrazov, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Problème de Tchebotarov. Nous donnons une solution complète du problème extrémal ayant posé par N.G. Tchebotarov vers 1920s, et nous établissons des formules explicites paramétriques pour les extrémales. ***Pour citer cet article : P. Tamrazov, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

In 1929 Polya [8] discussed the extremal problem earlier posed by Tchebotaröv. We formulate it in the well-known equivalent form: *among all univalent conformal mappings* f *of the unit disk* $K := \{z \in \mathbb{C}\colon |z| < 1\}$ *of the complex plane* $\mathbb{C}$ *into the plane* $\mathbb{C}$ *punctured at a finite number of fixed points* $a_1, \dots, a_m \in \mathbb{C} \setminus \{0\}$, *with* $f(0) = 0$, *find* f *for which the functional* $|f'(0)|$ *achieves its maximal value.*

The first essential results in the Tchebotaröv's problem were obtained by Lavrentiev [6,7] and Grötzsch [3] in 1930. Later Goluzin [1] and [2, p. 152–157] further developed these investigations. Besides existence and uniqueness of the extremal function f, these authors established also some qualitative and structural properties of the extremal and the following functional-differential equation for it (see [6,7,3,1] and [2, pp. 152–157]):

$$\left(\frac{zf'(z)}{f(z)}\right)^2 = \frac{p(f(z))}{q(f(z))}, \tag{1}$$

where $p(w) := \prod_{j=1}^{m}(a_j - w)$, and q is a polynomial on $w \in \mathbb{C}$ of the degree $m-1$ with $q(0) = \prod_{j=1}^{m} a_j$. Moreover, f is regular also on ∂K except a finite number of points, and the set $B := \overline{\mathbb{C}} \setminus f(K)$ is connected and is a union of a finite number of (open) analytic arcs and their endpoints. From here there follows that the domain $f(K)$ is admissible with respect to the quadratic differential

$$Q(w)\,\mathrm{d}w^2 := -\frac{q(w)}{w^2 p(w)}\,\mathrm{d}w^2 \tag{2}$$

E-mail addresses: proma@imath.kiev.ua, tamrazov@gyte.edu.tr (P. Tamrazov).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.08.007

(for terminology and main facts concerning theory of quadratic differentials, see [4]).

However, Eq. (1) contains $m-1$ complex-valued parameters (coefficients of q) whose values were unknown, and the problem of finding explicit formulae for the extremals of the Tchebotaröv's problem remained unsolved (see [2, p. 156; 9, p. 202]).

In the present work the author gives the general solution of the mentioned open problem – for any integer $m \geqslant 2$.

We emphasize that among numerous extremal problems generating quadratic differentials with a number of arbitrarily distributed fixed poles, it is the first case when the extremals are found in explicit form. And our methods enable also to solve a series of other extremal problems of the analogous nature.

Without any loss of generality we assume that all points $a_1, \dots, a_m$ in $\mathbb{C} \setminus \{0\}$ (cf. [7]) and $a_{m+1} = \infty$ are (simple) poles of $Q(w)\,\mathrm{d}w^2$. Such a point collection $\{a_j\} := \{a_j\}_{j=1}^{m+1}$ will be called *normalized*. All points of a normalized collection are endpoints of the set B (see [6,7]). We may consider B as a graph on $\overline{\mathbb{C}} \setminus \{0\}$ consisting of nodes of order one at all points a_j and only in them, nodes of orders $\nu_j + 2$ at all zeroes b_s of degrees $\nu_s \geqslant 1$, and only in them, and of all analytic trajectories of $Q(w)\,\mathrm{d}w^2$ (contained in B and ending at zeros or simple poles of $Q(w)\,\mathrm{d}w^2$) as edges of the graph. This curvilinear geometric graph is a tree, which we shall denote it by $L(\{a_j\})$. The total multiplicity of all zeroes of $Q(w)\,\mathrm{d}w^2$ equals $m-1$. Let now k be the number of *different* zeroes of $Q(w)\,\mathrm{d}w^2$.

Now we shall construct a class of *explicitly* defined functions containing all extremals for the Tchebotaröv's problem and only them. This class will be parametrized by means of special geometric rectilinear graphs defined in the complex plane. Let G be the class of all finite, undirected, connected, simple plane graphs Γ each of which satisfies the following conditions: (i) each edge γ of Γ is a rectilinear open interval in $\mathbb{C}$ of the length $|\gamma| > 0$, and these intervals mutually do not intersect each other, while nodes of the graph coincide with the endpoints of these intervals; (ii) Γ does not contain nodes of order 2 and cycles; (iii) the sum of lengths $|\gamma|$ of all intervals γ of the graph Γ equals π; (iv) the point $\zeta = 0$ is a node of Γ of order 1, and the edge of Γ incident to this point is contained in the real half-axis $\operatorname{Re}\zeta > 0$. Let $\operatorname{Supp}\Gamma$ denote the closure in $\mathbb{C}$ of the geometric union of all edges of the graph $\Gamma \in G$. Starting at the node 0, let us run along Γ in the direction in which the complementary to Γ domain $\mathbb{C} \setminus (\operatorname{Supp}\Gamma)$ remains on the left. Such a pass of Γ will be called *natural*. For every point ζ on an edge $\gamma \in \Gamma$, let $r_1(\Gamma, \zeta)$ and $r_2(\Gamma, \zeta)$ denote the length of the pass respectively to the first and the second reaching the point ζ, while $r_1(\Gamma, 0) = 0$, $r_2(\Gamma, 0) = 2\pi$. Under a single such pass along an edge γ the growth of each of functions r_1 and r_2 equals $|\gamma|$. For every node v of the order $\tau(v)$, let $r_1(\Gamma, v), \dots, r_{\tau(v)}(\Gamma, v)$ denote the length until the first, ..., $\tau(v)$th pass of v. For every $\zeta \in \operatorname{Supp}\Gamma$ and all $j = 1, \dots, \tau(\zeta)$ let us denote $\varepsilon_{\Gamma,j}(\zeta) := \exp(\mathrm{i} r_j(\Gamma, \zeta))$.

Let Γ' be one more graph from G, and for every $\zeta' \in \operatorname{Supp}\Gamma'$ the objects $\tau'(\zeta')$ and $\varepsilon_{\Gamma',j}(\zeta')$ be defined exactly as analogous objects were defined for Γ and $\zeta \in \operatorname{Supp}\Gamma$. Then the graphs Γ and Γ' will be called *equivalent*, if there exists the isomorphism $\eta : \Gamma \to \Gamma'$ such that $\eta(0) = 0$ and for every node v of Γ we have

$$r_j\bigl(\Gamma', \eta(v)\bigr) = r_j(\Gamma, v) \quad \forall j = 1, \dots, \tau(v).$$

If graphs $\Gamma, \Gamma' \in G$ are equivalent, then for every $\zeta \in \operatorname{Supp}\Gamma$ there corresponds a uniquely defined $\zeta' \in \operatorname{Supp}\Gamma'$ for which

$$\varepsilon_{\Gamma,j}(\zeta) = \varepsilon_{\Gamma',j}(\zeta') \quad \forall j = 1, \dots, \tau(\zeta).$$

For a graph Γ, let $V(\Gamma)$ be the set of all its nodes of order 1, and $W(\Gamma)$ be the set of all other its nodes (of orders $\geqslant 3$). Let V be the set of all points $\varepsilon_{\Gamma,1}(p)$ $(\in T)$, when p runs through the set $V(\Gamma)$. Denote by W_v the set of all points $\varepsilon_{\Gamma,j}(v)$ $(\in T)$, when $v \in W(\Gamma)$ is fixed and j runs through the set of values $1, \dots, \tau(v)$. Denote also $W := \bigcup_{v \in W(\Gamma)} W_v$. Clearly the point $z = 1$ is contained in V.

With any fixed branch of the below integrand continuous at the set $\overline{K} \setminus (W \cup \{1\})$, for $z \in \overline{K}$ let us consider the function

$$f(z) := \int_0^z (\zeta - 1)^{-3} \Biggl(\prod_{\alpha \in V \setminus \{1\}} (\zeta - \alpha) \Biggr) \prod_{v \in W(\Gamma)} \Biggl(\prod_{\beta \in W_v} (\zeta - \beta)^{2 - \tau(v)} \Biggr)^{1/\tau(v)} \mathrm{d}\zeta. \tag{3}$$

We have $|f'(0)| = 1$. Let f_K denote the restriction of f to K.

For a fixed graph $\Gamma \in G$ under the above notations and assumptions, we get the following result.

Theorem 1. *The function f given by* (3) *is holomorphic and univalent in K, continuous in $\overline{K} \setminus \{1\}$, continuous in generalized sense* (*with respect to topology of $\overline{\mathbb{C}}$ in the image*) *on $\overline{K}$. For every point $\zeta_0 \in \Gamma$ the function f glues* *rational-analytically all points* $\varepsilon_{\Gamma,j}(\zeta_0)$ $(j = 1, \dots, \tau(\zeta_0))$ *into one point denoted by* $y(\zeta_0)$, *and* f *is continuously and meromorphically extendable into a neighbourhood of every point* $z \in \overline{K} \setminus W$ (*holomorphically for every* $z \neq 1$). *Moreover* $f(\overline{K}) = \overline{\mathbb{C}}$, $f(0) = 0$, $f(1) = \infty$, *and the function* f_K *is extremal in the Tchebotaröv's problem for the collection of all points* $a(p) := y(p)$ *where* p *runs over the whole set* $V(\Gamma)$. *The extremal function in this problem for the mentioned collection of points* $a(p)$ *is unique up to rotation of the disk* K *in the* z*-plane around the origin. The set of all simple poles of the quadratic differential* $Q(w)\,\mathrm{d}w^2$ *given by* (2) *is normalized and hence coincides with the set of all* $m+1$ *points* $a(p)$, *including* $y(0) = \infty$, *while the set of all zeroes of* $Q(w)\,\mathrm{d}w^2$ *coincides with the set of all points* $b(v) := y(v)$, *where* v *runs over all* k *points of the set* $W(\Gamma)$. *Each point* $a(p)$ (*including* $a(0) = \infty$) *is an endpoint of some single trajectory of* $Q(w)\,\mathrm{d}w^2$. *The boundary of the domain* $f(K)$ *with respect to* $\overline{\mathbb{C}}$ *is the union of* $m+k$ *trajectories of* $Q(w)\,\mathrm{d}w^2$, *their* $m+1$ *endpoints* $a(p)$ $(\forall p \in V(\Gamma))$ *and* k *points* $b(v)$ $(\forall v \in W(\Gamma))$.

Let $\Gamma \in G$ be the fixed graph from Theorem 1 with all related to it objects and notations (in particular, $a(p)$ for all $V(\Gamma)$). Denote $L(\{a(p)\}) =: \Gamma_*$. Then we get the following result.

Theorem 2. *The graph* Γ_* *is isomorphic to* Γ, *with the correspondence of the node* $\zeta = 0$ *of* Γ *to the node* $w = \infty$ *of* Γ_*, *and the pass of* Γ_* *in the direction in which the domain* $\overline{\mathbb{C}} \setminus \Gamma_*$ *remains on the left, corresponds to the pass of* Γ *in the natural direction. Then the length of every pass along* Γ_* *in the metric* $|Q^{1/2}\,\mathrm{d}w|$ *equals to the length of its preimage on* Γ *with respect to the natural length measuring on* Γ.

Thus the graphs Γ and Γ_* are isomorphic, equally oriented relative to their complementary (with respect to $\overline{\mathbb{C}}$) domains and isometric in the sense of Theorem 2 (this isometry being consistent with the isomorphism and the direction of pass). From the definitions we see that for the equivalent graphs $\Gamma', \Gamma'' \in G$ and related to them objects corresponding to each other in this equivalence (including objects of the form p, v, $\varepsilon_{\Gamma,1}(p)$, $\varepsilon_{\Gamma,j}(v)$, $V(\Gamma)$, $W(\Gamma)$ for these graphs), the objects V, W, $\tau(v)$, W_v, $a(p)$, $b(v)$, f of similar form coincide. Let $\widetilde{G}$ denote the factor-set of G with respect to the equivalence. For a graph $\Gamma \in G$, let $\{\Gamma\}$ denote the class of all graphs from G equivalent to Γ.

Let N denote the set of all normalized point collections.

Let $H : \widetilde{G} \to N$ be the mapping defined for each $\widetilde{\Gamma}$ as a collection $\{f(p)\}_{p \in V(\Gamma)}$, where f is the function (3) defined for arbitrary $\Gamma \in \widetilde{\Gamma}$ and the corresponding $V(\Gamma)$.

Theorem 3. *The class of all extremals of the Tchebotaröv's problem is parametrized by elements of the set* $\widetilde{G}$ *and a positive number* r, *and this parametrization is one-to-one correspondence*: (1) *to every element* $\widetilde{\Gamma} \in \widetilde{G}$ *there corresponds one* (*and only one*) *normalized collection of points for which the function* f_K *with* f *given by* (3) *and corresponding to each graph* $\Gamma \in \widetilde{\Gamma}$ *is extremal in the Tchebotaröv's problem*; *and here we have* $|f'(0)| = 1$; (2) *and conversely, for every point collection* $\{a_j\} \in N$ *there exists one and only one class* $\widetilde{\Gamma} \in \widetilde{G}$ *and the unique positive constant* r *such that* $H(\widetilde{\Gamma}) = \{a_j/r\}$ *and the function* rf_K *with* f *defined by* (3) *for arbitrary* $\Gamma \in \widetilde{\Gamma}$ *is extremal in the Tchebotaröv's problem for* $\{a_j\}$.

For $m = 2$ our results give that the restriction to K of the function

$$f(z) := \int_0^z \frac{(\zeta + \mathrm{e}^{\mathrm{i}\delta_2})(\zeta + \mathrm{e}^{-\mathrm{i}\delta_3})\,\mathrm{d}\zeta}{(\zeta - 1)^3[(\zeta^2 - 2\zeta\cos\delta_1 + 1)(\zeta + \mathrm{e}^{\mathrm{i}(\delta_2 - \delta_3)})]^{1/3}}$$

with any constants $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, under $\delta_1 + \delta_2 + \delta_3 = \pi$, is extremal in the Tchebotaröv's problem for the collection of points $a_1 = f(\mathrm{e}^{\mathrm{i}(\delta_1+\delta_2)})$, $a_2 = f(\mathrm{e}^{-\mathrm{i}(\delta_1+\delta_3)})$, $a_3 = f(1) = \infty$. For comparison mention that Kuz'mina [5] found the extremal for the case $m = 2$ as an implicit solution of a system of equations containing elliptic Jacobi functions.

References

[1] G. Golusin, Method of variations in the theory of conform representation. I, Mat. Sb. 19 (61) (1946) 203–236 (in Russian).

[2] G. Golusin, Geometric Theory of Functions of a Complex Variable, Amer. Math. Soc., Providence, RI, 1969 (transl. from Russian).

[3] H. Grōtzsch, Über ein Variationsproblem der konformen Abbildung, Ber. Ferh. Sächs. Akad. Wiss. Leipzig 82 (1930) 251–263.
[4] J. Jenkins, Univalent Functions and Conformal Mappings, Inostr. Lit., Moscow, 1962 (Russian transl.).
[5] G. Kuz'mina, Covering theorems for functions holomorphic and univalent within a disk, Dokl. Acad. Nauk SSSR 160 (1965) 25–28 (in Russian).
[6] M. Lavrentieff, Sur un probleme de maximum dans la representation conforme, C. R. Acad. Sci. Paris 191 (1930) 827–829.
[7] M. Lavrentyev, On the theory of conformal mappings, Trudy Fiz.-Mat. Inst. Akad. Nauk SSSR 5 (1934) 159–246 (in Russian).
[8] G. Polya, Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete, III, S.-B. Preuss. Akad. Wiss. Berlin (1929) 55–62.
[9] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.