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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 715–718
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Mathematical Analysis/Functional Analysis

On some problems related to Berezin symbols

Mubariz T. Karaev

Department of Mathematics, Faculty of Arts and Sciences, Suleyman Demirel University, 32260 Isparta, Turkey

Received 28 March 2005; accepted 5 April 2005

Presented by Peter Lax

Abstract

The following problem was formulated by Zorboska [Proc. Amer. Math. Soc. 131 (2003) 793–800]: It is not known if the Berezin symbols of a bounded operator on the Bergman space $L_a^2(\mathbf{D})$ must have radial limits almost everywhere on the unit circle. In this Note we solve this problem in the negative, showing that there is a concrete class of diagonal operators for which the Berezin symbol does not have radial boundary values anywhere on the unit circle. A similar result is also obtained in case of the Hardy space $H^2(\mathbf{D})$ over the unit disk $\mathbf{D}$. Moreover, we give an alternative proof to the famous theorem of Beurling on z-invariant subspaces in the Hardy space $H^2(\mathbf{D})$, using the concepts of reproducing kernels and Berezin symbols. ***To cite this article: M.T. Karaev, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Quelques problèmes liés aux symboles de Berezin. Le problème suivant est formulé par Zorboska [Proc. Amer. Math. Soc. 131 (2003) 793–800] : les symboles de Berezin d'un opérateur borné sur l'espace de Bergman $L_a^2(D)$ ont-ils nécessairement des limites radiales presque partout sur le cercle unité ? Dans cet article, nous donnons une réponse négative à cette question en exhibant une classe concrète d'opérateurs diagonaux pour lesquels une telle limite n'existe en aucun point du cerle unité. Nous obtenons un résultat semblable dans le cas des espaces de Hardy $H^2(D)$ sur le dique unité D. De plus nous donnons une nouvelle preuve, utilisant les notions de noyaux reproduisants et de symboles de Berezin, du célèbre théorème de Beurling concernant les sous-espaces z-invariants de $H^2(D)$. ***Pour citer cet article : M.T. Karaev, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

A functional Hilbert space is a collection $\mathcal{H}$ of complex-valued functions on some set Ω such that $\mathcal{H}$ is a Hilbert space with respect to the usual vector operations on functions and which has the property that point evaluations are continuous (i.e., for each $\lambda \in \Omega$, the map $f \to f(\lambda)$ is continuous linear functional on $\mathcal{H}$). Prototypical functional

E-mail address: garayev@fef.sdu.edu.tr (M.T. Karaev).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.04.021

Hilbert spaces are the Hardy space $H^2(\mathbf{D})$ and the Bergman space $L^2_a(\mathbf{D})$ (for the definition and basic properties of these spaces see [12]).

If $\mathcal{H}$ is a functional Hilbert space, the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda\colon \lambda \in \Omega\}$is called the reproducing kernel of $\mathcal{H}$. It is well known (see, for instance [7], Problem 37) if $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$ on Ω, then the reproducing kernel of $\mathcal{H}$ is given by

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z). \tag{1}$$

It is easy to verify that the reproducing kernels of $H^2(\mathbf{D})$ and $L^2_a(\mathbf{D})$ are given by $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$ and $k_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^2}$, respectively.

For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For Aa bounded linear operator A on $\mathcal{H}$, the function $\tilde{A}$ defined on Ω by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ for $\lambda \in \Omega$ is the Berezin symbol of A. It is clear that the function $\tilde{A}$is bounded by the numerical radius of the operator A. It is also easy to see that on the most familiar functional Hilbert spaces, including the spaces $H^2(\mathbf{D})$and $L^2_a(\mathbf{D})$, the Berezin symbol uniquely determines the operator (i.e., $\tilde{A}(\lambda) = \widetilde{B}(\lambda)$ for all λ implies $A = B$), see for instance, [11,1,2,5]; for more general case, see [4], Theorem 1.1.1. Thus the Berezin symbol of a bounded operator contains a lot of information about the operator. Zorboska in [13] formulated the following natural and fundamental problem: it is not known if the Berezin transform of a bounded operator on $L^2_a(\mathbf{D})$ must have radial limits almost everywhere on the unit circle.

The present Note solves this problem negatively. Namely, we indicate a concrete class of diagonal operators on the Bergman space $L^2_a(\mathbf{D})$ for which the Berezin symbol does not have radial boundary values anywhere on the unit circle $\mathbf{T}$ (Theorem 2.2). A similar result is also proved in the Hardy space $H^2(\mathbf{D})$. Moreover, we give an alternative proof to the classical theorem of Beurling on z-invariant subspaces in $H^2(\mathbf{D})$, using the techniques of reproducing kernels and Berezin symbols.

2. Boundary behavior of Berezin symbols of operators on the Bergman and Hardy spaces

The Berezin symbol of an operator Aon the Bergman space has an explicit formula given by

$$\tilde{A}(\lambda) = \left(1 - |\lambda|^2\right)^2 \sum_{m,n=0}^{\infty} \sqrt{(n+1)(m+1)} \langle Ae_n, e_m \rangle \bar{\lambda}^n \lambda^m, \tag{2}$$

for $z \in \mathbf{D}$, where the functions $e_n(z) = \sqrt{n+1} z^n$, $n = 0, 1, \ldots,$ form the standard orthonormal basis for the Bergman space. Before giving our results, we note the following:

Definition 2.1. The sequence $\{a_n\}_{n=0}^{\infty}$ of the complex numbers a_n is *Abel convergent* (written (A) convergent) to a if the limit

$$\lim_{t \to 1^-} (1-t) \sum_{n=0}^{\infty} a_n t^n = a$$

exists.

Let S_- and S_-^∞ denote the set of all non-Abel convergent sequences of numbers and the set of all bounded non-Abel convergent sequences, respectively. It should be noticed that the existence of a bounded sequence, being not Abel summable, is obvious, since the Abel method and therefore each 'discrete (matrix-) Abel method' is regular and cannot sum each bounded sequence. Thus the set S_-^∞ is non-empty.

Let us consider the difference equation

$$(n+1)a_n - na_{n-1} = x_n, \quad n = 0, 1, 2, \ldots \tag{3}$$

(we put $a_{-1} \stackrel{\text{def}}{=} 0$). Elementary calculus shows that the solution of (3) is

$$a_n = \frac{1}{n+1}\sum_{i=0}^{n} x_i, \quad n \geqslant 0.$$

Consequently, for any given $\{x_n\} \in S_-^\infty$ the sequence $\{a_n\}_{n\geqslant 0} \stackrel{\text{def}}{=} \{\frac{1}{n+1}\sum_{i=0}^{n} x_i\}_{n\geqslant 0}$ is bounded and $\{(n+1)a_n - na_{n-1}\}_{n\geqslant 0} \in S_-^\infty$.

The following is our main result in this section, which solves Zorboska's problem in the negative.

Theorem 2.2. *Let $\{a_n\}_{n\geqslant 0}$ be a bounded sequence of numbers such that $\{(n+1)a_n - na_{n-1}\}_{n\geqslant 0} \in S_-$, and let $D_{\{a_n\}}$ be a diagonal operator with respect to the standard orthonormal basis $e_n(z) = \{\sqrt{n+1}\, z^n\}_{n\geqslant 0}$ of the Bergman space $L_a^2(\mathbf{D})$. Then the Berezin symbol $\widetilde{D}_{\{a_n\}}$ of the operator $D_{\{a_n\}}$ has no radial limits anywhere on the unit circle $\mathbf{T}$.*

Proof. By setting $A = D_{\{a_n\}}$ in the formula (2), we have

$$\widetilde{D}_{\{a_n\}}(\lambda) = \left(1 - |\lambda|^2\right)^2 \sum_{m=0}^{\infty} (m+1)a_m |\lambda|^{2m}$$

for all $\lambda \in \mathbf{D}$. Hence simple calculus shows that

$$\widetilde{D}_{\{a_n\}}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{m=0}^{\infty} \left[(m+1)a_m - ma_{m-1}\right] |\lambda|^{2m}, \tag{4}$$

$\lambda \in \mathbf{D}$ (i.e., $\widetilde{D}_{\{a_n\}}$ is a radial function, $\widetilde{D}_{\{a_n\}}(\lambda) = \widetilde{D}_{\{a_n\}}(|\lambda|)$). Since by condition of theorem, $\{(m+1)a_m - ma_{m-1}\}$ is not (A)-convergent sequence, it follows from (4) that the Berezin symbol $\widetilde{D}_{\{a_n\}}$ of the operator $D_{\{a_n\}}$ has no radial limits anywhere on the unit circle $\mathbf{T}$. The proof is completed. □

Our next result shows that there exist linear bounded operators on the Hardy space $H^2(\mathbf{D})$ such that their Berezin symbols have no radial boundary values on a unit circle.

Theorem 2.3. *Let $\{a_n\}_{n\geqslant 0} \in S_-^\infty$ be any sequence. Then the Berezin symbol of the diagonal operator $D_{\{a_n\}} z^n = a_n z^n$, $n \geqslant 0$, where $\{z^n\}_{n\geqslant 0}$ is a standard orthonormal basis of the Hardy space $H^2(\mathbf{D})$, has no radial limits anywhere on the unit circle $\mathbf{T}$.*

Proof. Simple calculus shows that

$$\widetilde{D}_{\{a_n\}}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}, \quad \lambda \in \mathbf{D}, \tag{5}$$

(i.e., $\widetilde{D}_{\{a_n\}}$ is a radial function, $\widetilde{D}_{\{a_n\}}(\lambda) = \widetilde{D}_{\{a_n\}}(|\lambda|)$). Considering that $\{a_n\}$ is not (A) convergent sequence, it follows from (5) that $\widetilde{D}_{\{a_n\}}(\lambda)$ has no radial limits anywhere on a unit circle, which completes the proof. □

Other applications of the formula (5) can be found in [9,8,10].

3. A new proof of Beurling's theorem

Let S be the unilateral shift operator on the Hardy space $H^2(\mathbf{D})$ defined by $(Sf)(z) = zf(z)$. The invariant subspaces of S are characterized by following famous Beurling's theorem.

Theorem 3.1 (Beurling [3]). *Every non-trivial invariant subspace of the shift operator S on the Hardy space $H^2(\mathbf{D})$ has the form $\theta H^2(\mathbf{D})$ for some inner function θ.*

In this section we shall give an alternative proof to the Beurling's theorem. Our proof uses the technique of reproducing kernels and Berezin symbols. We believe that our proof is much simpler (both technically and ideologically) and shorter than the original.

Proof of Theorem 3.1. Let $E \subset H^2(\mathbf{D})$ be a non-trivial (i.e., $\{0\} \neq E \neq H^2(\mathbf{D})$) invariant subspace of the shift operator S, that is $zE \subset E$. It is clear that $k_\lambda^E(z) \stackrel{\text{def}}{=} P_E k_\lambda(z)$, where $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$ is the reproducing kernel of $H^2(\mathbf{D})$, is the reproducing kernel of E. Since $(1-\bar{\lambda}z)\, k_\lambda^E(z)$is the reproducing kernel for the subspace $E \ominus zE$and $\dim(E \ominus zE) = 1$, by considering (1), one has $(1-\bar{\lambda}z)k_\lambda^E(z) = \overline{\theta(\lambda)}\theta(z)$ for some $\theta \in E \ominus zE$ with $\|\theta\| = 1$, and hence

$$\left(1-|\lambda|^2\right)k_\lambda^E(\lambda) = \left|\theta(\lambda)\right|^2, \quad \lambda \in \mathbf{D}. \tag{6}$$

On the other hand, it is not hard to check that

$$\left(1-|\lambda|^2\right)k_\lambda^E(\lambda) = \widetilde{P}_E(\lambda), \quad \lambda \in \mathbf{D}. \tag{7}$$

From (6) and (7), we obtain that

$$\widetilde{P}_E(\lambda) = \left|\theta(\lambda)\right|^2, \quad \lambda \in \mathbf{D}. \tag{8}$$

We shall now prove that θ is inner function. For this, by virtue of (8), it suffices to prove that

$$\widetilde{P}_E(\zeta) = 1 \quad \text{a.e. on } \mathbf{T}. \tag{9}$$

The proof of (9) is the same as the proof of Theorem 2.1 in [6] and we omit it.

Thus θ is an inner function. It is easy to verify that

$$\widetilde{P}_{\theta H^2(\mathbf{D})}(\lambda) = \left|\theta(\lambda)\right|^2, \quad \lambda \in \mathbf{D}. \tag{10}$$

Now it follows from (8) and (10) that $\widetilde{P}_E(\lambda) = \widetilde{P}_{\theta H^2(\mathbf{D})}(\lambda)$ for all $\lambda \in \mathbf{D}$, which implies (see [11,4]) that $P_E = P_{\theta H^2(\mathbf{D})}$, and hence $E = \theta H^2(\mathbf{D})$. Theorem 3.1 is proved. □

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