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Dynamical Systems/Ordinary Differential Equations

On sub-harmonic bifurcations

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Abstract

Under fairly general hypotheses, we investigate by elementary methods the structure of the p-periodic orbits of a family h_u of transformations near (u_0, x_0) when $h_{u_0}(x_0) = x_0$ and $\mathrm{d}h_{u_0}(x_0)$ has a simple eigenvalue which is a primitive p-th root of unity. ***To cite this article: M. Chaperon et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur les bifurcations sous-harmoniques. Sous des hypothèses très générales, nous étudions par des méthodes élémentaires la structure de l'ensemble des orbites de période p d'une famille h_u de transformations au voisinage de (u_0, x_0) lorsque $h_{u_0}(x_0) = x_0$ et que $\mathrm{d}h_{u_0}(x_0)$ a une valeur propre simple racine primitive p-ième de l'unité. ***Pour citer cet article : M. Chaperon et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Hypothèses et préliminaires

On se donne deux variétés banachiques U, V et une famille locale C^k, $k \geqslant 2$, d'applications de V dans lui-même dépendant du paramètre $u \in U$, c'est-à-dire une application $h : (U \times V, (u_0, x_0)) \to V$ de classe C^k. Si $k = \infty$ ou ω (C^ω signifie « analytique réel »), on pose $k - 1 := k$. On note $\tilde{h}(u, x) := (u, h_u(x))$ le *déploiement associé à* h. Étant donné un entier $p \geqslant 2$, on fait les hypothèses suivantes :

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H1 On a $h_{u_0}(x_0) = x_0$; autrement dit, (u_0, x_0) est un point fixe de $\tilde{h}$. On pose $E := T_{x_0}V$ et $L_0 := \mathrm{d}_{x_0}h_{u_0} : E \to E$.

H2 Si $p = 2$, L_0 admet $\rho = -1$ pour valeur propre simple : le sous-espace propre associé est une droite D, et -1 n'est pas dans le spectre $\operatorname{Spec} \dot{L}_0$ de l'endomorphisme $\dot{L}_0$ de E/D induit par L_0. *On suppose en outre* $1 \notin \operatorname{Spec} \dot{L}_0$.

H2′ Pour $p \geqslant 3$, le complexifié de L_0 admet une racine primitive p-ième ρ de l'unité pour valeur propre simple. Par conséquent, si $P \subset E$ désigne le 2-plan L_0-invariant engendré par les parties réelles et imaginaires des vecteurs propres correspondants, le spectre $\operatorname{Spec} \dot{L}_0$ du complexifié de l'endomorphisme $\dot{L}_0$ de E/P induit par L_0 ne contient ni ρ, ni $\bar{\rho}$. *On suppose en outre que* $\operatorname{Spec} \dot{L}_0$ *ne contient aucune racine p-ième de l'unité.*

On a donc $1 \notin \operatorname{Spec} L_0$, d'où, par le théorème des fonctions implicites, la

Proposition 0.1. *Au voisinage de (u_0, x_0), les points fixes de $\tilde{h}$ forment le graphe $x = \varphi(u)$ d'une application φ de classe C^k.*

Voici une autre conséquence du théorème des fonctions implicites :

Proposition 0.2. *Pour $p = 2$ (resp. > 2), il existe un unique germe $\alpha : (U, u_0) \to \mathbf{R}$ (resp. $\mathbf{C}$) d'application continue tel que $\alpha(u)$ soit valeur propre de $\mathrm{d}_{\varphi(u)}h_u$ et que $\alpha(u_0) = \rho$; de plus, α est de classe C^{k-1}.*

Nous pouvons énoncer notre dernière hypothèse :

H3 La différentielle $\mathrm{d}_{u_0}\alpha : T_{u_0}U \to \mathbf{R}$ (resp. $\mathbf{C}$) est surjective.

Notre résultat porte sur les *points périodiques de période p* des h_u, c'est-à-dire sur les $x \in V$ tels que $h_u^p(x) = x$; il est commode d'exprimer cela en disant que (u, x) est un point périodique de période p de $\tilde{h}$.

Théorème. *Sous ces hypothèses, les points périodiques de période p de $\tilde{h}$ forment, au voisinage de (u_0, x_0), la réunion de deux sous-variétés modelées sur U :*

(i) *la sous-variété C^k des points fixes de $\tilde{h}$, autrement dit le graphe W_1 de φ ;*

(ii) *une sous-variété W_p de classe C^m, où*

$$m := \begin{cases} k-1 & \text{si } p = 2, \\ \min\{[(k-2)/2], p-3\} & \text{pour } p > 3, \text{ d'où } m = p-3 \text{ si } k = \infty \text{ ou } \omega. \end{cases}$$

L'intersection $W_1 \cap W_p$ est la sous-variété C^{k-1} formée des $(u, \varphi(u))$ avec $\alpha(u) = \rho$. Pour $p = 2$ (resp. $p > 2$), elle est de codimension 1 (*resp.* 2) aussi bien dans W_p que dans W_1, *de sorte que l'ouvert $W_p \setminus W_1$ de W_p formé des points « vraiment périodiques » de période p de $\tilde{h}$ rencontre tout voisinage de (u_0, x_0) ; en outre, la sous-variété $W_p \setminus W_1$ est C^{k-1}.*

L'espace tangent $T_{(u_0,x_0)}(W_1 \cap W_p)$ est bien sûr $\{(\delta u, \mathrm{d}_{u_0}\varphi(\delta u)) \colon \mathrm{d}_{u_0}\alpha(\delta u) = 0\}$. Pour $m > 0$,

$$T_{(u_0,x_0)}W_p = \begin{cases} T_{(u_0,x_0)}(W_1 \cap W_p) \oplus (\{0\} \times D) & \text{si } p = 2, \\ T_{(u_0,x_0)}(W_1 \cap W_p) \oplus (\{0\} \times P) & \text{sinon.} \end{cases}$$

Remarque. Apparaît donc en (u_0, x_0) une famille à $\dim(U)$ paramètres d'orbites de période p. Le cas $p = 2$ était sûrement bien connu, au moins en dimension finie. Il n'en va pas vraiment de même du cas $p > 2$, même si les *langues d'Arnol'd* sont les projections dans U de nos variétés W_p dans le cas particulier des familles génériques à 2 paramètres. Dans la définition de m pour $p > 2$, nous ne savons pas si $[(k-2)/2]$ est optimal mais $p-3$ l'est, comme le montre l'exemple suivant : pour $U = V = \mathbf{C}$, la famille polynomiale $h_u(z) := (\rho + u)z - \bar{z}^{p-1}$ satisfait à nos hypothèses avec $u_0 = x_0 = 0$, $\varphi(u) = 0$ et $\alpha(u) = \rho + u$; la sous-variété W_p est donnée par $u = \bar{z}^{p-1}/z$ (en effet, ce sont des points périodiques et ils forment une surface C^{p-3} distincte du graphe de φ) ; elle n'est donc pas C^{p-2}, et cet exemple est typique de ce qui arrive en général.

1. Hypotheses and preliminaries

Consider two Banach manifolds U, V and a local C^k family, $k \geqslant 2$, of mappings of V into itself with parameter $u \in U$, i.e. a C^k map $h:(U \times V, (u_0, x_0)) \to V$. If $k = \infty$ or ω (C^ω means 'real analytic'), we set $k - 1 := k$. Let $\tilde{h}(u, x) := (u, h_u(x))$ (*unfolding associated to* h). Given an integer $p \geqslant 2$, assume the following:

H1 We have $h_{u_0}(x_0) = x_0$; in other words, (u_0, x_0) is a fixed point of $\tilde{h}$. Let $E := T_{x_0}V$ and $L_0 := \mathrm{d}_{x_0}h_{u_0} : E \to E$.
H2 If $p = 2$, L_0 admits $\rho = -1$ as a simple eigenvalue: the associated eigenspace is a line D, and -1 does not lie in the spectrum $\operatorname{Spec} \dot{L}_0$ of the endomorphism $\dot{L}_0$ of E/D induced by L_0. *Assume, moreover,* $1 \notin \operatorname{Spec} \dot{L}_0$.
H2′ For $p \geqslant 3$, the complexified $L_{0,\mathbf{C}}$ of L_0 admits a primitive p-th root ρ of unity as a simple eigenvalue. Therefore, if $P \subset E$ denotes the L_0-invariant 2-plane generated by the real and imaginary parts of the corresponding eigenvectors, the spectrum $\operatorname{Spec} \dot{L}_0$ of the complexified of the endomorphism $\dot{L}_0$ of E/P induced by L_0 contains neither ρ, nor $\bar{\rho}$. *Assume that, moreover,* $\operatorname{Spec} \dot{L}_0$ *does not contain any* p*-th root of unity.*

In particular, we have $1 \notin \operatorname{Spec} L_0$, hence, by the implicit function theorem,

Proposition 1.1. *Near* (u_0, x_0)*, the fixed points of* $\tilde{h}$ *form the graph of a* C^k *map* φ.

Here is another consequence of the implicit function theorem, proven in the sequel:

Proposition 1.2. *For* $p = 2$ (*resp.* > 2)*, there exists a unique continuous germ* $\alpha : (U, u_0) \to \mathbf{R}$ (*resp.* $\mathbf{C}$) *such that* $\alpha(u)$ *is an eigenvalue of* $\mathrm{d}_{\varphi(u)}h_u$ *and that* $\alpha(u_0) = \rho$*; moreover,* α *is* C^{k-1}.

We can now state our last hypothesis:

H3 The differential $\mathrm{d}_{u_0}\alpha : T_{u_0}U \to \mathbf{R}$ (resp. $\mathbf{C}$) is onto.

The *p-periodic points* of h_u are those $x \in V$ such that $h_u^p(x) = x$. This amounts to saying that (u, x) is a p-periodic point of $\tilde{h}$. Here comes our main result:

Theorem. *Under the above hypotheses, the* p*-periodic points of* $\tilde{h}$ *form, near* (u_0, x_0)*, the union of two submanifolds modelled on* U:
(i) *the* C^k *submanifold of fixed points of* $\tilde{h}$*, i.e. the graph* W_1 *of* φ;
(ii) *a* C^m *submanifold* W_p*, where*

$$m := \begin{cases} k-1 & \text{if } p = 2, \\ \min\{[(k-2)/2], p-3\} & \text{for } p > 3, \text{ hence } m = p-3 \text{ if } k = \infty \text{ or } \omega. \end{cases}$$

The intersection $W_1 \cap W_p$ *is the* C^{k-1} *submanifold consisting of those* $(u, \varphi(u))$ *with* $\alpha(u) = \rho$*. For* $p = 2$ (*resp.* $p > 2$)*, it has codimension* 1 (*resp.* 2) both in W_p and in W_1*, so that the open subset* $W_p \setminus W_1$ *of* W_p *containing the 'truly* p*-periodic' points of* $\tilde{h}$ *intersects every neighbourhood of* (u_0, x_0)*; moreover, the submanifold* $W_p \setminus W_1$ *is* C^{k-1} *even for* $p > 2$.

The tangent space $T_{(u_0,x_0)}(W_1 \cap W_p)$ *of course is* $\{(\delta u, \mathrm{d}_{u_0}\varphi(\delta u))\colon \mathrm{d}_{u_0}\alpha(\delta u) = 0\}$*. For* $m > 0$,

$$T_{(u_0,x_0)}W_p = \begin{cases} T_{(u_0,x_0)}(W_1 \cap W_p) \oplus (\{0\} \times D) & \text{if } p = 2, \\ T_{(u_0,x_0)}(W_1 \cap W_p) \oplus (\{0\} \times P) & \text{otherwise}. \end{cases}$$

Remark. For $p > 2$ and $U = \mathbf{R}^2$, when the family h_u is generic, the *Arnol'd tongues* are the projections in U of our W_p's. In the definition of m, we do not know whether $[(k-2)/2]$ is optimal but $p - 3$ is, as shown by the following example: for $U = V = \mathbf{C}$, the polynomial family $h_u(z) := (\rho + u)z - \bar{z}^{p-1}$ satisfies our hypotheses with $u_0 = x_0 = 0$, $\varphi(u) = 0$ and $\alpha(u) = \rho + u$; the submanifold W_p is given by $u = \bar{z}^{p-1}/z$ (indeed, those are periodic points and they form a C^{p-3} surface distinct from the graph of φ); it is not C^{p-2}, and this example is typical.

2. Proof of the theorem

Taking charts, we may assume that $V = E$, that $U = T_{u_0}U$ and that $(u_0, v_0) = (0, 0)$. Proposition 1.1 then follows from the implicit function theorem, applied to the equation $h_u(x) - x = 0$. The change of variables $(u, x) \mapsto (u, x - \varphi(u))$ enables us to replace the initial family by $(u, x) \mapsto h_u(\varphi(u) + x) - \varphi(u)$ or, in other words, to assume that $\varphi = 0$, i.e. $h_u(0) = 0$ for every u.

Proof of Proposition 1.2. If $p = 2$, choose any nonzero $\mathbf{v}_0 \in D$, any closed complementary subspace H of D in E, and identify E to $\mathbf{R} \times H$ by the isomorphism $(x, \mathbf{w}) \mapsto x\,\mathbf{v}_0 + \mathbf{w}$. Then, L_0 takes the form $\left(\begin{smallmatrix} -1 & b_0 \\ 0 & d_0 \end{smallmatrix}\right)$, where $d_0 : H \to H$ is a realisation of $\dot{L}_0$. Proposition 1.2 follows from the implicit function theorem, applied to the polynomial equation $F(L, \lambda, \mathbf{w}) := L(1, \mathbf{w}) - \lambda(1, \mathbf{w}) = 0$ at $(L_0, -1, 0)$; indeed, the partial $(\delta\lambda, \delta\mathbf{w}) \mapsto (b_0\delta\mathbf{w} - \delta\lambda, (d_0 + \mathrm{Id}_H)\delta\mathbf{w})$ of F with respect to $(\lambda, \mathbf{w})$ at $(L_0, -1, 0)$ is an automorphism because so is $d_0 + \mathrm{Id}_H$. Denoting the implicit function by $(\lambda(L), \mathbf{w}(L))$, we get $\alpha(u) = \lambda(\mathrm{d}_{\varphi(u)}h_u)$ and the associated eigenvector $\mathbf{v}(u) := (1, \mathbf{w}(\mathrm{d}_{\varphi(u)}h_u))$, which is a C^{k-1} function of u too.

For $p > 2$, we can submit $L_{0,\mathbf{C}}$ to the same treatment and get both α and a C^{k-1} map $\mathbf{v} : (U, u_0) \to E_{\mathbf{C}}$ such that each $\mathbf{v}(u)$ is a nonzero eigenvector of $(T_{\varphi(u)}h_u)_{\mathbf{C}} = Dh_u(0)_{\mathbf{C}}$ associated to the eigenvalue $\alpha(u)$. □

Proof of the theorem if $p = 2$**.** The map $\mathbf{R} \times H \ni (x, \mathbf{w}) \mapsto x\mathbf{v}(u) + \mathbf{w} \in E$ is, for u close to $0 = u_0$, an isomorphism depending C^{k-1} on u and conjugating $Dh_u(0)$ to an endomorphism of $\mathbf{R} \times H$ of the form $\left(\begin{smallmatrix} \alpha(u) & b(u) \\ 0 & d(u) \end{smallmatrix}\right)$. Now, since we have $\alpha(0) = -1 \notin \operatorname{Spec} d(0)$, we can kill $b(u)$ by the change of variables $(x, \mathbf{w}) \mapsto (x - b(u)(d(u) - \alpha(u)\,\mathrm{Id}_H)^{-1}\mathbf{w}, \mathbf{w})$. Composing those two changes of variables, we get a C^{k-1} family of isomorphisms $Q(u) : \mathbf{R} \times H \to E$ such that $Q(u)^{-1}Dh_u(0)Q(u) = \left(\begin{smallmatrix} \alpha(u) & 0 \\ 0 & d(u) \end{smallmatrix}\right)$.

Let $f_u(x, w) = f(u, x, w) \in \mathbf{R}$ and $g_u(x, w) = g(u, x, w) \in H$ denote the components of $Q(u)^*h_u^2(x, w) := Q(u) \circ h_u^2 \circ Q(u)^{-1}(x, w)$. We should solve the C^{k-1} system $f(u, x, w) = x$ and $g(u, x, w) = w$. As $\partial_w g_0(0, 0) = d(0)^2$ does not have 1 in its spectrum, the second equation defines a C^{k-1} implicit function $w = W(u, x) = W_u(x)$, which satisfies $W_u(0) = 0$ since $g_u(0, 0) = 0$, and $(W_u)'(0) = 0$ because $\partial_x g_u(0, 0) = 0$. Substituting in the first equation, we should solve

$$f_u\big(x, W_u(x)\big) = x, \tag{1}$$

whose solution $x = 0$ gives the fixed points. After factoring them out, (1) becomes

$$F(u, x) = 1, \tag{2}$$

where $F(u, x) = \int_0^1 Df_u(tx, W_u(tx))(1, (W_u)'(tx))\,\mathrm{d}t$ *is* C^{k-1}. Indeed, $Df_u(x, w)$ and $Dg_u(x, w)$ are C^{k-1} functions of (u, x, w), being the components of $Q(u)^{-1} \circ Dh_u^2(Q(u)(x, w)) \circ Q(u)$, and it follows that $(W_u)'(x) = (\mathrm{Id}_H - \partial_w g_u(x, W_u(x)))^{-1}\,\partial_x g_u(x, W_u(x))$ is a C^{k-1} function of (u, x).

Choose a decomposition $U = \mathbf{R} \times U_0$, $u = (\mu, \nu)$, such that $\partial_\nu\alpha(0) = 0$, hence $\partial_\mu\alpha(0) \neq 0$ by H3. As $(W_u)'(0) = 0$, we have $F(u, 0) = \partial_x f_u(0, 0) = \alpha(u)^2$ (hence the part about $W_1 \cap W_2$ in the theorem) and $\partial_\mu F(0, 0) = -2\partial_\mu\alpha(0) \neq 0$. Therefore, (2) defines a C^{k-1} implicit function $\mu = M(\nu, x)$ and W_2 is the graph $(\mu, w) = (M(\nu, x), W(M(\nu, x), \nu, x))$. Let us prove that this function is tangent to 0 at 0, hence the part about $T_{(u_0, x_0)}W_2$ in the theorem: as $DW(0) = 0$, we should show that $DM(0) = 0$, i.e. $\partial_{(\nu, x)}F(0) = 0$, i.e. $\partial_x F(0) = 0$, i.e. $\frac{1}{2}\partial_x^2 f(0) = 0$. Now, as $D^2h_0^2(0) = D^2h_0(0) \circ (Dh_0(0) \times Dh_0(0)) + Dh_0(0) \circ D^2h_0(0)$, denoting by $f_1(u, x, w)$ the first component of $Q(u)^*h_u(x, w)$, we do get $\partial_x^2 f(0) = \partial_x^2 f_1(0) - \partial_x^2 f_1(0) = 0$. □

Proof of the theorem for $p > 2$. We start again by a diagonalization result:

Lemma 2.1. *There exist a Banach space H and a C^{k-1} family of isomorphisms $Q(u) : \mathbf{C} \times H \to E$ such that the derivative of $Q(u)^* h_u = Q(u)^{-1} \circ h_u \circ Q(u)$ at 0 is $\begin{pmatrix} \alpha(u) & 0 \\ 0 & d(u) \end{pmatrix}$, $d(u) \in L(H)$.*

Proof. With the notation of the proof of Proposition 1.2, let H be any closed complementary subspace of the 2-plane P generated by $\Re \mathbf{v}(0)$ and $\Im \mathbf{v}(0)$. For u close to $0 = u_0$, the mapping $\mathbf{C} \times H \ni (x + \mathrm{i}y, \mathbf{w}) \mapsto x \Re \mathbf{v}(u) - y \Im \mathbf{v}(u) + \mathbf{w} \in E$ is an isomorphism depending C^{k-1} on u and conjugating $Dh_u(0)$ to an endomorphism of $\mathbf{C} \times H$ of the form $\begin{pmatrix} \alpha(u) & b(u) \\ 0 & d(u) \end{pmatrix}$. Now, we can kill $b(u)$ by the variable change $(z, \mathbf{w}) \mapsto (z + c(u)\mathbf{w}, \mathbf{w})$, $c(u) \in L(H, \mathbf{C})$, provided $\alpha(u)c(u) - c(u) \circ d(u) = b(u)$; as the hypothesis $\operatorname{Spec} d(0) \not\ni \rho$ implies that the spectrum $\operatorname{Spec} d(u)$ of the complex endomorphism ${}^t d(u) : c \mapsto c \circ d(u)$ of $L(H, \mathbf{C})$ does not contain $\alpha(u)$ for small u, this equation has the unique solution $c(u) = (\alpha(u) \operatorname{Id}_{L(H,\mathbf{C})} - {}^t d(u))^{-1} b(u)$, a C^{k-1} function of u. Composing our two variable changes, we do get a C^{k-1} family of isomorphisms $Q(u) : \mathbf{C} \times H \to E$ such that $Q(u)^{-1} \circ Dh_u(0) \circ Q(u) = \begin{pmatrix} \alpha(u) & 0 \\ 0 & d(u) \end{pmatrix}$. □

Denoting by $f_j(u, z, w) \in \mathbf{C}$ and $g_j(u, z, w) \in H$ the components of $Q(u)^* h_u^j(z, w)$, we have to solve the system of class C^{k-1} $f_p(u, z, w) = z$ and $g_p(u, z, w) = w$. As $\partial_w g_p(0, 0, 0) = d(0)^p$ does not have 1 in its spectrum, the second equation defines a C^{k-1} implicit function $w = W(u, z) = W_u(z)$ satisfying $W_u(0) = 0$ and (since $\partial_z f_p(u, 0) = 0$) $DW_u(0) = 0$. Thus, we should solve

$$f_p\big(u, z, W_u(z)\big) = z, \tag{3}$$

whose spurious solution $z = 0$ again gives the fixed points, but this time if we divide the left-hand side by z we get a function which may even be discontinuous. However, we can prove

Lemma 2.2. *Fix a decomposition $U = \mathbf{R}^2 \times U_0$, $u = (\mu, \nu)$, such that $\partial_\nu \alpha(0) = 0$ and therefore $\partial_\mu \alpha(0)$ is an isomorphism. Then, we have the following*:

(i) *The solutions of* (3) *near* $(0, 0)$ *form the union of* $\{z = 0\}$ *and the graph $\mu = M(\nu, z)$ of a continuous function, C^{k-1} in $\{z \neq 0\}$.*

(ii) *Therefore, near $(u_0, x_0) = (0, 0)$, the manifold W_p is the set of those $(u, Q(u)(z, w))$ such that $(\mu, w) = (M(\nu, z), \underline{W}(\nu, z))$, where $\underline{W}(\nu, z) := W(M(\nu, z), \nu, z)$ is continuous, C^{k-1} in $\{z \neq 0\}$.*

(iii) *All the statements of the theorem are true, except perhaps smoothness.*

Proof. In polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, we have $f_p(u, r\,\mathrm{e}^{\mathrm{i}\theta}, W_u(r\,\mathrm{e}^{\mathrm{i}\theta})) = r\,\mathrm{e}^{\mathrm{i}\theta} F(u, r, \theta)$, where

$$F(u, r, \theta) = \mathrm{e}^{-\mathrm{i}\theta} \int_0^1 \partial_{(z,w)} f_p\big(u, tr\,\mathrm{e}^{\mathrm{i}\theta}, W_u(tr\,\mathrm{e}^{\mathrm{i}\theta})\big)\big(\mathrm{e}^{\mathrm{i}\theta}, DW_u(tr\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}t$$

satisfies $F(u, 0, \theta) = \mathrm{e}^{-\mathrm{i}\theta} \partial_z f_p(u, 0, 0)\,\mathrm{e}^{\mathrm{i}\theta} = \alpha(u)^p$ since $DW_u(0) = 0$. After factoring out the fixed points $r\,\mathrm{e}^{\mathrm{i}\theta} = 0$, Eq. (3) to be solved becomes

$$F(u, r, \theta) = 1. \tag{4}$$

Now, we have that $\partial_\mu F(0, 0, \theta) = \partial_\mu \alpha^p(0)$, which is an isomorphism. Moreover, *we can make the crucial remark that F is C^{k-1}*. Indeed,

(a) $\partial_{(z,w)} f_j(u, z, w)$ and $\partial_{(z,w)} g_j(u, z, w)$ are the components of $Q(u)^{-1} \circ Dh_u^j(Q(u)(z, w)) \circ Q(u)$, therefore $\partial_{(z,w)} f_j$ and $\partial_{(z,w)} g_j$ are C^{k-1};

(b) and so is $\partial_z W : (u, z) \mapsto (\operatorname{Id}_H - \partial_w g_p(u, z, W_u(z)))^{-1} \circ \partial_z g_p(u, z, W_u(z))$.

It follows that the solutions of (4) near $\{(u,r)=(0,0)\}$ are given by a C^{k-1} implicit function $\mu=\widetilde{M}(\nu,r,\theta)$. As $F(u,0,\theta)=\alpha(u)^p$ is independent of θ, so is $\widetilde{M}(\nu,0,\theta)$, hence (i)–(ii) with $M(\nu,r\,\mathrm{e}^{\mathrm{i}\theta})=\widetilde{M}(\nu,r,\theta)$.

To prove (iii), first notice that now W_1 is defined by $z=0$ and that $F(u,0,\theta)=1$ writes $\alpha(u)^p=1$, which near $u=0$ is equivalent to $\alpha(u)=\rho$ since α is continuous and $\alpha(0)=\rho$. This establishes the characterization of $W_1\cap W_p$ in the theorem. Let us now prove that *if M is differentiable, then $DM(0,0)=0$*. Since $D\underline{W}(0,0)=0$, this will imply that (after our coordinate changes) $T_{(0,0)}W_p=\{(\mu,w)=(0,0)\}$, hence the formula for $T_{(u_0,x_0)}W_p$ in the theorem. As $\mathrm{d}\widetilde{M}(0,0,\theta)=\partial_\nu M(0,0)\,\mathrm{d}\nu+\partial_z M(0,0)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}r$ when the right-hand side is defined, we should prove for $p>3$ that $D\widetilde{M}(0,0,\theta)=0$ or, in other words, $\partial_{(\nu,r,\theta)}F(0,0,\theta)=0$. Since $\partial_{(\nu,\theta)}F(0,0,\theta)=0$, this means proving that $\partial_r F(0,0,\theta)=0$, i.e. $\frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\partial_z^2 f_p(0)(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}\theta})=0$. Now, differentiating $Q(0)^*h_0^p$ twice yields $\partial_z^2 f_p(0)(Z,Z)=\sum_0^{p-1}\rho^{-j-1}\partial_z^2 f_1(0)(\rho^j Z,\rho^j Z)$, which is 0, as $\partial_z^2 f_1(0)(Z,Z)=aZ^2+bZ\overline{Z}+c\overline{Z}^2$. □

This proves our theorem if $m=0$. Otherwise, to actually factor by z and get more smoothness, we cannot put directly the left-hand side of (3) into normal form. Instead, we use

Lemma 2.3. *If (u,z) is a solution of (3), then z is a p-periodic point of $\Phi_u : z\mapsto f_1(u,z,W_u(z))$.*

Proof. If $y=(z,w)$ is a p-periodic point of $Q(u)^*h_u$, so is $Q(u)^*h_u^j(y)$ for all $j\geqslant 1$, hence $g_j(u,y)=W(u,f_j(u,y))$ for all $j\in\mathbf{N}$. It follows at once by induction that $\Phi_u^j(z)=f_j(u,y)$ for all $j\in\{0,\dots,p\}$, which implies that $\Phi_u^p(z)=f_p(u,y)=z$ as required. □

Lemma 2.4. *Assuming $m>0$, let the decomposition $U=\mathbf{R}\times U_0$, $u=(\mu,\nu)$, be the same as in Lemma* 2.2. *Near $(0,0)\in U\times\mathbf{C}$, the set of those (u,z) such that z is a p-periodic point of Φ_u form the union of $\{z=0\}$ and the graph $\mu=M_1(\nu,z)$ of a C^m function. By Lemmas* 2.2 *and* 2.3, *we must have $M_1=M$, which shows that W_1 is C^m and completes the proof of the theorem.*

Proof. Note that, because of properties (a)–(b) in the proof of Lemma 2.2, both $\Phi_u(z)$ and $D\Phi_u(z)$ are C^{k-1} functions of (u,z). Therefore, Taylor's formula implies that $\Phi_u(z)=\alpha(u)z+T_2+\dots+T_{m+1}+T_{m+2}$ where T_j is a homogeneous polynomial of degree j in $z,\bar{z}$ whose coefficients are C^{k-j} functions of u for $j\leqslant m+1$ and C^{k-m-2} functions of u,z for $j=m+2$. Since we have $m\leqslant p-3$, we can make for $j=2,\dots,m+1$ a C^{k-j} change of coordinates of the form $z\mapsto z+c_j(u)\bar{z}^j$ so that in the end the coefficient of $\bar{z}^j$ in T_j is 0 for $1\leqslant j\leqslant m+1$, hence $\Phi_u(z)=z(a(u,z)+b(u,z)\bar{z}^{m+2}/z)$ with a,b of class C^{k-m-2}. Now, $F_1(u,z):=a(u,z)+b(u,z)\bar{z}^{m+2}/z$ is C^m like $\bar{z}^{m+2}/z$ since we have $m\leqslant k-m-2$. It follows that $h_u^j(z)=zF_j(u,z)$ with F_j of class C^m and $F_j(u,0)=\alpha(u)^j$. Therefore, the equation $F_p=1$ defines a C^m implicit function $\mu=M_2(\nu,z)$. As our changes of coordinates are C^m and tangent to the identity, the graph of M_2 is the graph of a C^m function M_1 in the original coordinates. □

Remark. Here are three reasons why we do not use a center manifold:
– We *cannot*, as $\mathrm{d}_{x_0}h_{u_0}$ may have other spectral values on the unit circle.
– Center manifolds of C^∞ or analytic maps are not C^∞ in general—see [1], where the same idea is used to show that a manifold of periodic orbits is C^∞ or analytic.
– Lemma 2.3 is trivial, whereas the center manifold theorem is not.

This work owes much to the dissertation [2] and to discussions with Shirley Bromberg.

References

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