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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 581–586
COMPTES RENDUS MATHEMATIQUE
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Algebraic Geometry

Del Pezzo surface fibrations obtained by blow-up of a smooth curve in a projective manifold

Toru Tsukioka

Institut Élie-Cartan, université H. Poincaré, Nancy 1, UMR 7502, BP 239, 54506 Vandoeuvre-lès-Nancy cedex, France

Received 27 January 2005; accepted 4 March 2005

Available online 7 April 2005

Presented by Jean-Pierre Demailly

Abstract

This Note is devoted to the study of the Fano manifolds X obtained by blow-up along a smooth curve C in a complex projective manifold Y. By the Mori theory, we can ensure the existence of an extremal contraction $\varphi : X \to Z$ different from the blow-up $\pi : X \to Y$. Here we give the complete classification of the corresponding pairs (Y, C) in the case where φ is a fiber type contraction of relative dimension 2, i.e. the general fibers of φ are del Pezzo surfaces. In Tsukioka (Thesis, Nancy University 1, 2005), the relative dimension 1 case is also considered. ***To cite this article: T. Tsukioka, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Fibrations en surfaces de del Pezzo obtenues par éclatement d'une courbe lisse dans une variété projective. Cette Note est consacrée à l'étude des variétés de Fano X obtenues par éclatement d'une courbe lisse C dans une variété projective complexe et lisse Y. D'après la théorie de Mori, on peut assurer l'existence d'une contraction extrémale $\varphi : X \to Z$ différente de l'éclatement $\pi : X \to Y$. Ici, on donne la classification complète des paires correspondantes (Y, C) dans le cas où φ est de type fibrant de dimension relative 2, c'est-à-dire quand les fibres générales de φ sont des surfaces de del Pezzo. Dans Tsukioka (thèse, université Nancy 1, 2005) le cas de dimension relative 1 est aussi étudié. ***Pour citer cet article : T. Tsukioka, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail address: tsukioka@iecn.u-nancy.fr (T. Tsukioka).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.03.004

Version française abrégée

Dans [2] les auteurs ont classifié les variétés complexes dont l'éclatée en un point est de Fano. Une extension naturelle est le problème suivant :

Problème. Soit X l'éclatée d'une variété projective complexe et lisse Y le long d'une courbe lisse C. Classifier les paires (Y, C) telles que X est de Fano (c'est-à-dire le fibré anticanonique $-K_X$ est ample).

Le but de cette Note est de donner la classification des paires (Y, C) lorsque X a une structure d'une fibration en surfaces de del Pezzo. On démontre le :

Théorème 0.1. *Soit $\pi : X \to Y$ l'éclatement d'une variété projective complexe et lisse de dimension $n \geqslant 4$ le long d'une courbe lisse C. On suppose que X a une contraction extrémale $\varphi : X \to Z$ telle que les fibres lisses de φ sont des surfaces de del Pezzo (φ est donc nécessairement équidimensionnelle). On suppose aussi que φ est élémentaire, c'est-à-dire $\rho(X) = \rho(Z) + 1$ (d'où $\rho(Y) = \rho(Z)$). Alors on est exactement dans l'un des cas suivants :*

(i) *Y est isomorphe à $\mathbb{P}^n$ et C est une droite ;*
(ii) *Y est isomorphe à Q_n une hypersurface quadrique de $\mathbb{P}^{n+1}$ et C est une conique (intersection de $n-1$ sections hyperplanes) ;*
(iii) *Y est une variété de del Pezzo et C est intersection complète de $n-1$ sections hyperplanes ;*
(iv) *(seulement pour $n = 4$) Y est isomorphe à $\mathbb{P}^4$ et C est intersection complète de trois quadriques.*

D'après [5], pour une variété de Fano Y on définit l'indice r_Y par :

$$r_Y := \max\bigl\{m \in \mathbb{N} \mid \text{il existe } H \in \operatorname{Pic}(Y) \text{ tel que } -K_Y \sim mH\bigr\}.$$

Le critère de Kobayashi–Ochiai montre que l'indice des variétés de Fano de dimension n est majoré par $n+1$ et que $\mathbb{P}^n$ (resp. Q_n) est la seule variété de Fano d'indice $n+1$ (resp. n). Les variétés de Fano d'indice $n-1$ sont dites *variétés de del Pezzo* et sont classifiées par [4].

Idée de la démonstration du Théorème 0.1

On note E le diviseur exceptionnel de π. Les fibres de $\pi_{|E} : E \to C$ sont isomorphes à $\mathbb{P}^{n-2}$ et la restriction de φ à une fibre de $\pi_{|E}$ est une surjection sur Z. D'où $\rho(Z) = 1$ et par hypothèse, on a aussi $\rho(Y) = 1$. On note $\mathcal{O}_Y(1)$ le générateur ample de $\operatorname{Pic}(Y) \simeq \mathbb{Z}$ et on pose $H := \pi^*\mathcal{O}_Y(1)$.

Soit f une courbe rationnelle minimale de la contraction φ. Puisque la fibre générale lisse S de φ est une surface de del Pezzo, on a soit : (i) S est isomorphe à $\mathbb{P}^2$ et f est une droite de $\mathbb{P}^2$, soit : (ii) S est isomorphe à Q_2 et f est une fibre de la projection naturelle $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, soit : (iii) S est une surface de del Pezzo non minimale et f est une (-1)-courbe dans une fibre de φ. Ainsi, la démonstration du théorème 0.1 se ramène à l'étude de ces trois cas. Le point crucial est l'égalité $H \cdot f = 1$. Elle se montre dans le cas (iii) par un argument complètement numérique alors que dans les cas (i) et (ii) on peut la montrer d'une manière plus géométrique en établissant d'abord l'autre égalité $E \cdot f = 1$.

Comme X est l'éclatée de Y le long d'une courbe, on a $K_X \sim \pi^*K_Y + (n-2)E \sim -r_Y H + (n-2)E$ où r_Y est l'indice de Y. L'égalité $H \cdot f = 1$ nous permet d'avoir : $-K_X \cdot f + (n-2)E \cdot f = r_Y$. On rappelle que $r_Y \leqslant n+1$ par [5]. Donc

(i) si $S \simeq \mathbb{P}^2$ $(-K_X \cdot f = 3)$, alors $r_Y = n+1$ c'est-à-dire $Y \simeq \mathbb{P}^n$;
(ii) si $S \simeq Q_2$ $(-K_X \cdot f = 2)$, alors $r_Y = n$ c'est-à-dire $Y \simeq Q_n$;
(iii) si S est une surface de del Pezzo non minimale $(-K_X \cdot f = 1)$, alors soit $r_Y = n-1$, et Y est une variété de del Pezzo, soit $n = 4$ et $r_Y = 5$, et donc $Y \simeq \mathbb{P}^4$.

La structure de la courbe C est facilement déterminée par des calculs numériques.

1. Introduction

In [2] the authors classified the complex manifolds whose blow-up at a point is Fano. As a next step it is natural to consider the following problem:

Problem. Let Y be a complex projective manifold. Let $\pi : X \to Y$ be the blow-up along a smooth curve C. Classify the pairs (Y, C) such that X is Fano (i.e. the anticanonical bundle $-K_X$ is ample).

Remark that by the list of [7], we can determine the pairs (Y, C) in dimension 3. For the toric case the classification is known in any dimension (see [9]).

By the Mori theory, we can ensure the existence of an extremal contraction $\varphi : X \to Z$ to a normal projective variety, different from the blow-up π. The purpose of this Note is to give the complete classification of the pairs (Y, C) in the case where the general fibers of φ are of dimension 2. Since $\dim \varphi^{-1}(z) \leqslant 2$ for all $z \in Z$ (see the argument of [2] Section 2), φ is equidimensional. If S is a smooth fiber of φ, S is a del Pezzo surface because $-K_S = (-K_X)_{|S}$ is ample. We will prove the following:

Theorem 1.1. *Let $\pi : X \to Y$ be the blow-up of a complex projective manifold of dimension $\geqslant 4$ along a smooth curve C. We assume that X has an extremal contraction $\varphi : X \to Z$ such that the general smooth fiber of φ is a del Pezzo surface* (*so φ is necessarily equidimensional*). *We assume also that φ is elemental, i.e. $\rho(X) = \rho(Z) + 1$* (*so $\rho(Y) = \rho(Z)$*). *Then we have exactly the one of the following*:

(i) *Y is isomorphic to $\mathbb{P}^n$ and C is a line*;
(ii) *Y is isomorphic to a smooth hyperquadric Q_n in $\mathbb{P}^{n+1}$ and C is a conic* (*intersection of $n-1$ hyperplane sections*);
(iii) *Y is a del Pezzo manifold and C is a complete intersection of $n-1$ hyperplane sections*;
(iv) (*$n = 4$ only*) *Y is isomorphic to $\mathbb{P}^4$ and C is a complete intersection of three quadrics in $\mathbb{P}^4$.*

Recall that for a Fano manifold Y the index r_Y is defined by:

$$r_Y := \max\{m \in \mathbb{N} \mid \text{ there exists } H \in \mathrm{Pic}(Y) \text{ such that } -K_Y \sim mH\}.$$

By definition, $r_{\mathbb{P}^n} = n + 1$ and $r_{Q_n} = n$. The Kobayashi–Ochiai's criterion [5] says that the index of the n-dimensional Fano manifolds is bounded above by $n + 1$ and that $\mathbb{P}^n$ (resp. Q_n) is the only Fano manifold whose index is equal to $n + 1$ (resp. n). A *del Pezzo manifold* is defined as a Fano manifold whose index is equal to $n - 1$, and is completely classified (see [4]).

2. Preliminary results on intersection numbers

The exceptional divisor of the blow-up $\pi : X \to Y$ will be denoted by E. For a point $a \in C$ let $E_a := \pi^{-1}(a) \simeq \mathbb{P}^{n-2}$. Since there exists a surjective map $\varphi_{|E_a} : E_a \to Z$, we get $\rho(Z) = 1$ and by assumption of the theorem, we have also $\rho(Y) = 1$. We denote $\mathcal{O}_Y(1)$ (resp. $\mathcal{O}_Z(1)$) the ample generator of $\mathrm{Pic}(Y)$ (resp. $\mathrm{Pic}(Z)$). We will use the notations $H := \pi^*\mathcal{O}_Y(1)$ and $L := \varphi^*\mathcal{O}_Z(1)$. Let f be a *minimal* rational curve of the contraction φ, i.e. f is a rational curve in a fiber of φ such that for any curve Γ contracted by φ we have $-K_X \cdot f \leqslant -K_X \cdot \Gamma$. Since the smooth fiber S of φ is a del Pezzo surface, we are in one of the following cases:

(i) φ is a $\mathbb{P}^2$-fibration, i.e. $S \simeq \mathbb{P}^2$. So f is a line in a fiber of φ and we have $-K_X \cdot f = 3$;
(ii) φ is a Q_2-fibration, i.e. $S \simeq Q_2$. So f is a fiber of the natural projection $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and we have $-K_X \cdot f = 2$;
(iii) φ is a non minimal del Pezzo surface fibration, i.e. S is a del Pezzo surface with (-1)-curves. In this case f is a (-1)-curve in a general fiber and we have $-K_X \cdot f = 1$.

In each case, let T be the deformation space of the rational curve f and (f) denote the corresponding point in T. For $t \in T$ the corresponding rational curve is written by f_t.

Lemma 2.1. *If φ is a $\mathbb{P}^2$-fibration or a Q_2-fibration, we have $E \cdot f = 1$.*

Proof. Suppose $E \cdot f \geqslant 2$. For a general fiber S of φ, $E \cap S$ is an irreducible and reduced curve of degree at least 2 in $S \simeq \mathbb{P}^2$ (resp. of bidegree (k,k) with $k \geqslant 2$ in $S \simeq Q_2$) if φ is a $\mathbb{P}^2$-fibration (resp. Q_2-fibration). So we can take $(f_0) \in T$ such that $\mathrm{Card}(E \cap f_0) \geqslant 2$. Let x_1, x_2 be two distinct points of $E \cap f_0$. Let $a_i := \pi(x_i)$ $(i = 1, 2)$. A simple dimension estimate ensures the existence of a curve $B \subset T$ such that for all $t \in B$ the corresponding rational curve f_t meets the two fibers $E_{a_i} := \pi^{-1}(a_i)$ $(i = 1, 2)$. So the union $F := \bigcup_{t \in B} f_t$ is a ruled surface having two exceptional curves $F \cap E_{a_1}$ and $F \cap E_{a_2}$, a contradiction. Therefore $E \cdot f = 1$. □

In the case of a non minimal del Pezzo surface fibration, it seems difficult to show the equality $E \cdot f = 1$, because $E \cdot f \geqslant 2$ does not imply immediately the existence of $(f_0) \in T$ such that $\mathrm{Card}(E \cap f_0) \geqslant 2$. We will use another lemma:

Lemma 2.2. *If φ is a non minimal del Pezzo surface fibration, we have $H \cdot f = 1$.*

Proof. Let e be a line in a fiber of $\pi_{|E} : E \to C$. Then $L \cdot e = H \cdot f$ because these two integers are equal to the order of the cyclic group $\mathrm{Pic}(X)/\mathbb{Z}[H] \oplus \mathbb{Z}[L]$ (see the proof of [8] Theorem 5.1). We now write $L \equiv x\pi^*(-K_Y) - yE$ with $x, y \in \mathbb{Q}$. Note that $y = L \cdot e (= H \cdot f) \in \mathbb{N}$. So it is sufficient to show $y = 1$. Since Z is normal and dominated by $E_a \simeq \mathbb{P}^{n-2}$, we conclude that Z is isomorphic to $\mathbb{P}^{n-2}$ using the following three results: (1) if a normal projective variety Z is dominated by a projective space, then Z has at most quotient singularities (see [3] Theorem 4.2); (2) if the extremal contraction $\varphi : X \to Z$ is equidimensional and if Z has at most quotients singularities, then Z is smooth ([1] Proposition 1.3); (3) if a smooth projective variety Z is dominated by $\mathbb{P}^m$, $Z \simeq \mathbb{P}^m$ ([6] Theorem 4.1). In particular $\mathcal{O}_Z(1)^{n-2} = 1$ and so $L^{n-2} \equiv S$. Since $L^{n-1} \equiv 0$, we have $L^{n-1} \cdot E = 0$, $L^{n-1} \cdot \pi^*(-K_Y) = 0$ and $L^{n-2} \cdot (-K_X)^2 = (-K_S)^2 =: a$ where $1 \leqslant a \leqslant 7$. Since $L \equiv x\pi^*(-K_Y) - yE$, we get

$$\begin{cases} (n-1)xy^{n-2}d - y^{n-1}\delta = 0, \\ x^{n-1}m - y^{n-1}d = 0, \\ x^{n-2}m - 2(n-2)y^{n-2}d - (n-2)^3 y^{n-3}(xd) + (n-2)^2 y^{n-2}\delta = a \end{cases}$$

where $m := (\pi^*(-K_Y))^n = (-K_Y)^n$, $d := (-1)^n(\pi^*(-K_Y)) \cdot E^{n-1} = -K_Y \cdot C$ and $\delta := (-1)^n E^n = \deg N_{C/Y}$ are integers. From these relations we have

$$xd = y\delta/(n-1) \tag{1}$$

and

$$x^{n-1}m = y^{n-1}d, \tag{2}$$

$$x^{n-2}m = a + y^{n-2}\bigl(2(n-2)d - (n-2)^2\delta/(n-1)\bigr). \tag{3}$$

By (2) we have $(y/x)^n = m/d$. On the other hand (2) and (3) imply

$$\frac{y}{x} = y \cdot \frac{x^{n-2}m}{x^{n-1}m} = \frac{a + y^{n-2}(2(n-2)d - (n-2)^2\delta/(n-1))}{y^{n-2}d}.$$

Thus

$$\big(a/y^{n-2}+2(n-2)d-(n-2)^2\delta/(n-1)\big)^{n-1}=md^{n-2}\in\mathbb{N}.$$

It follows that

$$k:=a/y^{n-2}-(n-2)^2\delta/(n-1)\in\mathbb{Z}. \tag{4}$$

Finally,

$$(n-1)a=y^{n-2}\big((n-1)k+(n-2)^2\delta\big). \tag{5}$$

Recall that $1\leqslant a\leqslant 7$ and that $y\in\mathbb{N}$. Hence we have immediately $y=1$ if $n\geqslant 6$. To show $y=1$ for $n=4$ and 5, some more arithmetical arguments will be needed. We only explain the case of $n=4$: we suppose that $y\neq 1$. By (5), we have $3a\equiv 0 \pmod{y^2}$. If $y=2$, we get $1=-K_X\cdot f=r_YH\cdot f-2(E\cdot f)=2(r_Y-E\cdot f)$, contradiction. So, $y=3$ and $a=3$ or 6. We have $1=-K_X\cdot f=3r_Y-2E\cdot f$, so there are three possibilities $(r_Y,E\cdot f)=(1,1),\ (3,4)$ or $(5,7)$. For example, $(r_Y,E\cdot f)=(3,4)$ is not possible: in this case, Y is a del Pezzo manifold. We get $m:=(-K_Y)^4=(3H)^4=3^4h$ where $h:=H^4=1,2,3,4$ or 5 by a result of [4]. By (1), $xd=\delta$ and by (2), $3x^3h=d$. Let $x=p/q$ (the integers p and q are supposed to be coprime). We have $3p^3h=dq^3$, so q^3 divides $3h$. But since $1\leqslant h\leqslant 5$, this implies that $q=1$. Hence $x=p\in\mathbb{Z}$. We obtain $3x^3h=d$ so $d\equiv 0 \pmod 3$ and finally $\delta=xd\equiv 0 \pmod 3$. By (4), $a/27=k+(4\delta/3)\in\mathbb{Z}$, but since $a=3$ or 6, this is a contradiction. Similarly we can easily exclude the other possibilities. □

3. Proof of Theorem 1.1

We prove three propositions (Proposition 3.1, 3.2 and 3.3) which imply Theorem 1.1.

Proposition 3.1. *If the general fiber of φ is isomorphic to $\mathbb{P}^2$, Y is isomorphic to $\mathbb{P}^n$ and C is a line.*

Proof. By Lemma 2.1, for a general point $z\in Z$, $\widetilde{C}_z:=E\cap\varphi^{-1}(z)$ is a line in $\varphi^{-1}(z)\simeq\mathbb{P}^2$. So $\varphi_{|E}:E\to Z$ is a $\mathbb{P}^1$-fibration and C is isomorphic to $\mathbb{P}^1$. Thus $E\simeq\mathbb{P}^1\times\mathbb{P}^{n-2}$. Since Z is normal, the morphism $\varphi_{|E}:E\to Z$ (resp. $\pi_{|E}:E\to C$) coincides with the natural projection $\mathbb{P}^1\times\mathbb{P}^{n-2}\to\mathbb{P}^{n-2}$ (resp. $\mathbb{P}^1\times\mathbb{P}^{n-2}\to\mathbb{P}^1$). In particular, $\varphi_{|E_a}:E_a\to Z$ and $\pi_{|\widetilde{C}_z}:\widetilde{C}_z\to C$ are isomorphisms. So we have $L\cdot e=1$ and $\pi_*f=\pi_*\widetilde{C}_z=C$. Since $\mathrm{Pic}(X)=\mathbb{Z}[H]\oplus\mathbb{Z}[E]$, there exist $a,\ b\in\mathbb{Z}$ such that $L\sim aH+bE$. Taking the intersection numbers with e and f we get $a(H\cdot f)=1$. Thus $a=H\cdot f=1$. But now, $3=-K_X\cdot f=r_YH\cdot f-(n-2)E\cdot f=r_Y-(n-2)$ and so $r_Y=n+1$. By [5], $Y\simeq\mathbb{P}^n$. Since $\pi_*f=C$, we have: $\mathcal{O}_Y(1)\cdot C=H\cdot f=1$ which implies that C is a line in $Y\simeq\mathbb{P}^n$. □

Proposition 3.2. *If the general fiber of φ is isomorphic to Q_2, Y is isomorphic to Q_n and C is a conic.*

Proof. The same type of arguments as in the case of a $\mathbb{P}^2$-fibration shows that the curve $\widetilde{C}_z:=E\cap\varphi^{-1}(z)$ is numerically equivalent to $2f$ and $H\cdot f=1$. Now $2=-K_X\cdot f=r_YH\cdot f-(n-2)E\cdot f=r_Y-(n-2)$ and so $r_Y=n$. By [5] again, $Y\simeq Q_n$. On the other hand $\mathcal{O}_Y(1)\cdot C=H\cdot\widetilde{C}_z=H\cdot(2f)=2$. This implies that C is a conic in $Y\simeq Q_n$. □

Proposition 3.3. *If the general fiber of φ is a del Pezzo surface with (-1)-curves, then either Y is a del Pezzo manifold and C is a complete intersection of $n-1$ members of $|\mathcal{O}_Y(1)|$ or Y is isomorphic to $\mathbb{P}^4$ and C is a complete intersection of three quadrics in $\mathbb{P}^4$.*

Proof. By Lemma 2.2 we have $-K_X \cdot f = r_Y H \cdot f - (n-2)E \cdot f = r_Y - (n-2)E \cdot f$. Since $-K_X \cdot f = 1$ and $r_Y \leqslant n+1$ by [5] we get $E \cdot f \leqslant n/(n-2)$. This implies that: if $n \geqslant 5$, $E \cdot f = 1$, and if $n = 4$, $E \cdot f = 1$ or 2. If $E \cdot f = 1$, Y is a del Pezzo manifold because $r_Y = n-1$. Take $n-1$ general members $L_1, \ldots, L_{n-1}$ of the linear system $|L|$. Let $M_i := \pi(L_i)$ $(i = 1, \ldots, n-1)$. Since $L \sim H - E$, $M_i = \pi(L_i) \in |\mathcal{O}_Y(1)|$. We have $C \subset M_1 \cap \cdots \cap M_{n-1}$. On the other hand, $0 = H \cdot L^{n-1} = H \cdot (H-E)^{n-1} = H^n + (-1)^{n-1} H \cdot E^{n-1} = \mathcal{O}_Y(1)^n - \mathcal{O}_Y(1) \cdot C$ so $\mathcal{O}_Y(1)^n = \mathcal{O}_Y(1) \cdot C$, and therefore $C \equiv \mathcal{O}_Y(1)^{n-1}$. Thus, $C = M_1 \cap \cdots \cap M_{n-1}$ is a complete intersection. If $n = 4$ and $E \cdot f = 2$, by Kobayashi-Ochiai's criterion, $Y \simeq \mathbb{P}^4$ because $r_Y = 5$. Take three general members L_1, L_2, L_3 of $|L|$, and let $Q_i := \pi(L_i)$ $(i = 1, 2, 3)$. Since $L \sim 2H - E$ (because $E \cdot f = 2$), $Q_i = \pi(L_i) \in |\mathcal{O}_{\mathbb{P}^4}(2)|$. We have $C \subset Q_1 \cap Q_2 \cap Q_3$. On the other hand, $0 = H \cdot L^3 = H \cdot (2H - E)^3 = 8H^4 - H \cdot E^3 = 8(\mathcal{O}_{\mathbb{P}^4}(1))^4 - \mathcal{O}_{\mathbb{P}^4}(1) \cdot C$ hence $C \equiv (\mathcal{O}_{\mathbb{P}^4}(2))^3$. Thus, $C = Q_1 \cap Q_2 \cap Q_3$ is a complete intersection. □

Acknowledgements

The author would like to express his gratitude to Professor Frédéric Campana for encouragement and many valuable discussions.

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