



Partial Differential Equations

An isoperimetric inequality for the principal eigenvalue of the Laplacian with drift

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Abstract

We generalize the classical Rayleigh–Faber–Krahn inequality to the case of the Dirichlet Laplacian with a drift. We also solve some optimization problems for the principal eigenvalue of the operator $-\Delta + v \cdot \nabla$ in a fixed domain with a control of the drift v in L^∞ . **To cite this article:** *F. Hamel et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005)*.

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Résumé

Une inégalité isopérimétrique pour la valeur propre principale du laplacien avec transport. Nous généralisons l'inégalité classique de Rayleigh–Faber–Krahn au cas du laplacien Dirichlet avec un terme de transport. Nous résolvons également des problèmes d'optimisation pour la valeur propre principale de l'opérateur $-\Delta + v \cdot \nabla$ dans un domaine fixé et avec un contrôle L^∞ sur le terme de transport v . **Pour citer cet article :** *F. Hamel et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005)*.

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On note $\lambda_1(\Omega)$ la première valeur propre du laplacien $-\Delta$ avec la condition au bord de Dirichlet dans un domaine borné $\Omega \subset \mathbb{R}^n$. L'inégalité classique de Rayleigh–Faber–Krahn [7,10,11,17] affirme que

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) = |\Omega|^{-2/n} \alpha_n^{2/n} j_{n/2-1,1}^2, \quad (1)$$

où $|\Omega|$ est la mesure de Lebesgue de Ω , $\alpha_n = \pi^{n/2} / \Gamma(n/2 + 1)$ la mesure de Lebesgue de la boule unité euclidienne de \mathbb{R}^n et $j_{m,1}$ le premier zéro strictement positif de la fonction de Bessel J_m . On désignera toujours par Ω^* la boule

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euclidienne centrée en 0 et telle que $|\Omega^*| = |\Omega|$. Beaucoup d'autres problèmes d'optimisation pour les autres valeurs propres de $-\Delta$ ont été résolus par la suite.

On s'intéresse ici à un problème de minimisation de la première valeur propre d'un opérateur *non auto-adjoint*, le laplacien Dirichlet auquel on ajoute un terme de transport. On note \mathcal{C} l'ensemble des ouverts connexes bornés de \mathbb{R}^n de classe $C^{2,\alpha}$ pour un $\alpha \in (0, 1)$. Si $\Omega \in \mathcal{C}$ et $v: \Omega \rightarrow \mathbb{R}^n$ est une fonction mesurable, on dit que $v \in L^\infty(\Omega, \mathbb{R}^n)$ si $|v|$ (la norme euclidienne de v) appartient à $L^\infty(\Omega)$, et on note $\|v\|_\infty$ la norme L^∞ de $|v|$. Pour tout $\Omega \in \mathcal{C}$ et tout $v \in L^\infty(\Omega, \mathbb{R}^n)$, on note $\lambda_1(\Omega, v)$ la première valeur propre de l'opérateur $L = -\Delta + v \cdot \nabla$ avec condition au bord de Dirichlet. On a le résultat suivant :

Théorème 0.1. *Pour tous $n \geq 1$, $\Omega \in \mathcal{C}$, $\tau \geq 0$ et $v \in L^\infty(\Omega, \mathbb{R}^n)$ vérifiant $\|v\|_\infty \leq \tau$, on a*

$$\lambda_1(\Omega, v) \geq \lambda_1(\Omega^*, \tau e_r), \quad (2)$$

où $e_r(x) = x/|x|$ pour tout $x \neq 0$. De plus, l'égalité n'a lieu dans (2) que si, à translation près, $\Omega = \Omega^*$ et $v = \tau e_r$, c'est-à-dire s'il existe $x_0 \in \mathbb{R}^n$ tel que $(\Omega, v) = (x_0 + \Omega^*, \tau e_r(\cdot - x_0))$.

Le Théorème 0.1 généralise la première conjecture de Rayleigh au cas du laplacien Dirichlet avec transport. En corollaire, on obtient une inégalité de type Faber–Krahn pour ces opérateurs : il existe une fonction $F_n: (0, +\infty) \times [0, +\infty) \rightarrow (0, +\infty)$, donnée par $F_n(m, \tau) = \lambda_1(B_{(m/\alpha_n)^{1/n}}, \tau e_r)$, telle que

$$\lambda_1(\Omega, v) \geq F_n(|\Omega|, \|v\|_\infty)$$

pour tout $\Omega \in \mathcal{C}$ et tout $v \in L^\infty(\Omega, \mathbb{R}^n)$ (on désigne par B_r la boule euclidienne centrée en 0 et de rayon r). On prouve aussi que $F_1(m, \tau) \sim \tau^2 e^{-\tau m/2}$ quand $\tau \rightarrow +\infty$, que $F_n(m, \tau) > F_1(2(m/\alpha_n)^{1/n}, \tau)$ pour tous $n \geq 2$, $m > 0$ et $\tau \geq 0$, et que $\log F_n(m, \tau) \sim -\tau(m/\alpha_n)^{1/n}$ quand $\tau \rightarrow +\infty$.

Un résultat intermédiaire dans la preuve du Théorème 0.1, intéressant en lui-même, concerne l'optimisation de $\lambda_1(\Omega, v)$ lorsque Ω est fixé et que la norme L^∞ de v est contrôlée. Posons, pour tout $\tau \geq 0$,

$$\underline{\lambda}(\Omega, \tau) = \inf_{\|v\|_\infty \leq \tau} \lambda_1(\Omega, v) \quad \text{et} \quad \bar{\lambda}(\Omega, \tau) = \sup_{\|v\|_\infty \leq \tau} \lambda_1(\Omega, v).$$

Théorème 0.2. *On fixe $\Omega \in \mathcal{C}$ (de classe $C^{2,\alpha}$ pour un $0 < \alpha < 1$) et $\tau \geq 0$. Il existe un unique $\underline{v} \in L^\infty(\Omega)$ (resp. $\bar{v} \in L^\infty(\Omega)$) avec $\|\underline{v}\|_\infty \leq \tau$ (resp. $\|\bar{v}\|_\infty \leq \tau$) tel que*

$$\underline{\lambda}(\Omega, \tau) = \lambda_1(\Omega, \underline{v}) \quad (\text{resp. } \bar{\lambda}(\Omega, \tau) = \lambda_1(\Omega, \bar{v})).$$

De plus, $|\underline{v}| = \tau$ (resp. $|\bar{v}| = \tau$) p.p. dans Ω , la fonction propre associée $\underline{\varphi}$ (resp. $\bar{\varphi}$) est $C^{2,\alpha}(\bar{\Omega})$ et $\underline{v} \cdot \nabla \underline{\varphi} = -\tau |\nabla \underline{\varphi}|$ (resp. $\bar{v} \cdot \nabla \bar{\varphi} = +\tau |\nabla \bar{\varphi}|$) dans Ω . La fonction $\underline{\varphi}$ (resp. $\bar{\varphi}$) est solution du problème non linéaire :

$$-\Delta \underline{\varphi} - \tau |\nabla \underline{\varphi}| = \underline{\lambda}(\Omega, \tau) \underline{\varphi} \quad \text{dans } \Omega \quad (3)$$

$$(\text{resp. } -\Delta \bar{\varphi} + \tau |\nabla \bar{\varphi}| = \bar{\lambda}(\Omega, \tau) \bar{\varphi} \quad \text{dans } \Omega). \quad (4)$$

Enfin, si ψ est une fonction $C^{2,\alpha}(\bar{\Omega})$ telle que $\psi > 0$ dans Ω , $\psi = 0$ sur $\partial\Omega$ et si $\mu \in \mathbb{R}$ est tel que (3) (resp. (4)) soit satisfaite avec ψ et μ au lieu de $\underline{\varphi}$ et $\underline{\lambda}(\Omega, \tau)$ (resp. $\bar{\varphi}$ et $\bar{\lambda}(\Omega, \tau)$), alors $\psi = \underline{\varphi}$ (resp. $\psi = \bar{\varphi}$) à une constante multiplicative près, et $\mu = \underline{\lambda}(\Omega, \tau)$ (resp. $\mu = \bar{\lambda}(\Omega, \tau)$). De plus, si $\Omega = B_r$, alors $\underline{v} = \tau e_r$, $\bar{v} = -\tau e_r$ et les fonctions propres correspondantes $\underline{\varphi}$ et $\bar{\varphi}$ sont radiales et décroissantes dans B_r .

La preuve du Théorème 0.2 repose sur le principe du maximum et le lemme de Hopf, tandis que celle du Théorème 0.1 utilise le Théorème 0.2 et un procédé non standard de symétrisation des fonctions [8].

1. Introduction and main result

Call $\lambda_1(\Omega)$ the first eigenvalue of the Laplacian $-\Delta$ with Dirichlet boundary condition, in a bounded domain of \mathbb{R}^n . The classical Rayleigh–Faber–Krahn inequality states that

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) = |\Omega|^{-2/n} \alpha_n^{2/n} j_{n/2-1,1}^2, \tag{5}$$

where $|\Omega|$ is the Lebesgue measure of Ω , $j_{m,1}$ the first positive zero of the Bessel function J_m and $\alpha_n = \pi^{n/2} / \Gamma(n/2 + 1)$ is the Lebesgue measure of the unit open Euclidean ball B_1 in \mathbb{R}^n . From now on, Ω^* denotes the ball B_R of \mathbb{R}^n with center 0 and radius R such that $|\Omega^*| = |\Omega|$ ($R = (|\Omega|/\alpha_n)^{1/n}$). Furthermore, equality (5) holds only when Ω is a ball. This result was first conjectured by Rayleigh ([17], 1894/1896) for $n = 2$, and proved independently by Faber ([7], 1923) and Krahn ([10], 1925) for $n = 2$, and by Krahn for all n ([11], 1926).

After these works, many papers dealt with the questions of bounding or optimizing the other eigenvalues, or some functions of the eigenvalues, of the Dirichlet Laplacian [1,2,5,6,9,12,14–16]. Other boundary conditions have been considered (see for instance [18,19] for the optimization of the second eigenvalue of the Neumann Laplacian). Lastly, for the Rayleigh clamped plate problem (to minimize the first eigenvalue of Δ^2 in a domain with given measure with Dirichlet–Neumann boundary conditions), it is known that in dimensions 2 [13] and 3 [3] the minimum is achieved for, and only for, the ball.

All above problems concern self-adjoint operators. Here, we address the question of minimizing the first eigenvalue of the *non-self-adjoint* Dirichlet Laplacian *with a drift* in a domain. By ‘domain’, we mean an open connected subset of \mathbb{R}^n , and we denote by \mathcal{C} the set of all bounded domains of \mathbb{R}^n which are of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. If $\Omega \in \mathcal{C}$ and $v : \Omega \rightarrow \mathbb{R}^n$ is measurable, $|v|$ will denote the Euclidean norm of v , and we say that $v \in L^\infty(\Omega, \mathbb{R}^n)$ if $|v| \in L^\infty$, and denote by $\|v\|_\infty$ the L^∞ norm of $|v|$. For any $\Omega \in \mathcal{C}$ and $v \in L^\infty(\Omega, \mathbb{R}^n)$, let $\lambda_1(\Omega, v) > 0$ denote the principal Dirichlet eigenvalue of $L = -\Delta + v \cdot \nabla$.

Our main result is the following:

Theorem 1.1. *For any $n \geq 1$, $\Omega \in \mathcal{C}$, $\tau \geq 0$ and $v \in L^\infty(\Omega, \mathbb{R}^n)$ satisfying $\|v\|_\infty \leq \tau$, one has*

$$\lambda_1(\Omega, v) \geq \lambda_1(\Omega^*, \tau e_r), \tag{6}$$

where $e_r(x) = x/|x|$ for all $x \neq 0$. Moreover, equality holds in (6) only when, up to translation, $\Omega = \Omega^*$ and $v = \tau e_r$, namely when there is $x_0 \in \mathbb{R}^n$ such that $(\Omega, v) = (x_0 + \Omega^*, \tau e_r(\cdot - x_0))$.

Theorem 1.1 can then be viewed as a natural extension of the first Rayleigh conjecture to the Dirichlet Laplacian with drift in any dimension n .

Since the principal eigenvalue of an elliptic operator is decreasing with respect to the inclusion of domains [4], an immediate consequence of Theorem 1.1 is that, for any $m > 0$ and $\tau \geq 0$,

$$\inf_{\Omega \in \mathcal{C}, |\Omega| \leq m, \|v\|_\infty \leq \tau} \lambda_1(\Omega, v) = \lambda_1(B, \tau e_r),$$

where $B = B_{(m/\alpha_n)^{1/n}}$ (that is $|B| = m$). Furthermore, the infimum is achieved for, and only for, $\Omega = B$ and $v = \tau e_r$ (up to translation).

Another immediate corollary of Theorem 1.1 is the following Faber–Krahn type inequality for the Dirichlet Laplacian with drift: there exists a function $F_n : (0, +\infty) \times [0, +\infty) \rightarrow (0, +\infty)$, defined by $F_n(m, \tau) = \lambda_1(B_{(m/\alpha_n)^{1/n}}, \tau e_r)$, such that

$$\lambda_1(\Omega, v) \geq F_n(|\Omega|, \|v\|_\infty)$$

for all $\Omega \in \mathcal{C}$ and $v \in L^\infty(\Omega, \mathbb{R}^n)$. With one-dimensional comparison arguments (for radial solutions), one can also prove that $F_1(m, \tau) \sim \tau^2 e^{-\tau m/2}$ when $\tau \rightarrow +\infty$, that $F_n(m, \tau) > F_1(2(m/\alpha_n)^{1/n}, \tau)$ for all $n \geq 2$, $m > 0$ and $\tau \geq 0$, and that $\log F_n(m, \tau) \sim -\tau(m/\alpha_n)^{1/n}$ when $\tau \rightarrow +\infty$. However, giving an equivalent for $F_n(m, \tau)$ when τ is large and $n \geq 2$ is an open question.

2. Minimization and maximization of the first eigenvalue in a fixed domain

The first step in the proof of Theorem 1.1, which has its own interest, is the optimization of $\lambda_1(\Omega, v)$ when Ω is a fixed domain and the L^∞ norm of v is controlled. Namely, for any $\tau \geq 0$, set

$$\underline{\lambda}(\Omega, \tau) = \inf_{\|v\|_\infty \leq \tau} \lambda_1(\Omega, v) \quad \text{and} \quad \bar{\lambda}(\Omega, \tau) = \sup_{\|v\|_\infty \leq \tau} \lambda_1(\Omega, v).$$

It turns out that this optimization problem also has a unique solution, which can also be explicitly found in the case where the domain is a ball:

Theorem 2.1. *Let Ω be a domain in \mathcal{C} (of class $C^{2,\alpha}$ for some $0 < \alpha < 1$) and let $\tau \geq 0$ be fixed. There exists a unique vector field $\underline{v} \in L^\infty(\Omega)$ (resp. $\bar{v} \in L^\infty(\Omega)$) with $\|\underline{v}\|_\infty \leq \tau$ (resp. $\|\bar{v}\|_\infty \leq \tau$) such that*

$$\underline{\lambda}(\Omega, \tau) = \lambda_1(\Omega, \underline{v}) \quad (\text{resp. } \bar{\lambda}(\Omega, \tau) = \lambda_1(\Omega, \bar{v})).$$

Moreover, $|\underline{v}| = \tau$ (resp. $|\bar{v}| = \tau$) a.e. in Ω , the corresponding principal eigenfunction $\underline{\varphi}$ (resp. $\bar{\varphi}$) is of class $C^{2,\alpha}(\bar{\Omega})$ and $\underline{v} \cdot \nabla \underline{\varphi} = -\tau |\nabla \underline{\varphi}|$ (resp. $\bar{v} \cdot \nabla \bar{\varphi} = +\tau |\nabla \bar{\varphi}|$) in Ω . The function $\underline{\varphi}$ (resp. $\bar{\varphi}$) then solves the following nonlinear problem

$$-\Delta \underline{\varphi} - \tau |\nabla \underline{\varphi}| = \underline{\lambda}(\Omega, \tau) \underline{\varphi} \quad \text{in } \Omega \tag{7}$$

(resp.

$$-\Delta \bar{\varphi} + \tau |\nabla \bar{\varphi}| = \bar{\lambda}(\Omega, \tau) \bar{\varphi} \quad \text{in } \Omega). \tag{8}$$

Moreover, if ψ is a function of class $C^{2,\alpha}(\bar{\Omega})$ such that $\psi > 0$ in Ω , $\psi = 0$ on $\partial\Omega$ and if $\mu \in \mathbb{R}$ is such that (7) (resp. (8)) holds with ψ and μ instead of $\underline{\varphi}$ and $\underline{\lambda}(\Omega, \tau)$ (resp. $\bar{\varphi}$ and $\bar{\lambda}(\Omega, \tau)$), then $\psi = \underline{\varphi}$ (resp. $\psi = \bar{\varphi}$) up to multiplication, and $\mu = \underline{\lambda}(\Omega, \tau)$ (resp. $\mu = \bar{\lambda}(\Omega, \tau)$).

Assume now that $\Omega = B$ is a ball of center 0, and let $\tau \geq 0$ be fixed. Then $\underline{v} = \tau e_r$, $\bar{v} = -\tau e_r$ and the corresponding eigenfunctions $\underline{\varphi}$ and $\bar{\varphi}$ are radially decreasing in \bar{B} .

A consequence of Theorems 1.1 and 2.1 is that, for all $\tau \geq 0$ and $\Omega \in \mathcal{C}$, one has $\underline{\lambda}(\Omega, \tau) \geq \underline{\lambda}(\Omega^*, \tau)$, and equality holds only when Ω is a ball.

3. Some comments and related optimization problems

One can now wonder what happens for other optimization problems with analogous constraints. For a fixed domain $\Omega \in \mathcal{C}$, if we drop the condition $\|v\|_\infty \leq \tau$, one can easily prove that

$$\inf_{v \in L^\infty(\Omega, \mathbb{R}^n)} \lambda_1(\Omega, v) = \inf_{\tau \geq 0} \underline{\lambda}(\Omega, \tau) = 0, \quad \sup_{v \in L^\infty(\Omega, \mathbb{R}^n)} \lambda_1(\Omega, v) = \sup_{\tau \geq 0} \bar{\lambda}(\Omega, \tau) = +\infty.$$

The first assertion (about the infimum) follows at once from the monotonicity property of the first eigenvalue with respect to the inclusion of domains, and from the fact that $\lambda_1(B, \tau e_r(\cdot - x_0)) = F_n(|B|, \tau) \rightarrow 0$ as $\tau \rightarrow +\infty$, for any given ball centered at x_0 . For the proof of the other assertion, it is straightforward to check that $\lambda_1(\Omega, \tau e_1) = \tau^2/4 + \lambda_1(\Omega, 0) \rightarrow +\infty$ as $\tau \rightarrow +\infty$, where $e_1(x) = (1, 0, \dots, 0)$ for all x .

Let now $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be fixed, let Ω vary and define, for $m > 0$,

$$\underline{\underline{\lambda}}(v, m) = \inf_{\Omega \in \mathcal{C}, |\Omega|=m} \lambda_1(\Omega, v|_\Omega) = \inf_{\Omega \in \mathcal{C}, |\Omega| \leq m} \lambda_1(\Omega, v|_\Omega) \quad \text{and} \quad \bar{\bar{\lambda}}(v, m) = \sup_{\Omega \in \mathcal{C}, |\Omega|=m} \lambda_1(\Omega, v|_\Omega).$$

From some results in [4], it is easy to check that $\bar{\bar{\lambda}}(v, m) = +\infty$. On the other hand, $\underline{\underline{\lambda}}(v, m) \geq F_n(m, \|v\|_\infty)$. However, unlike $\underline{\lambda}(\Omega, \tau)$ or $\bar{\lambda}(\Omega, \tau)$, the infimum in $\underline{\underline{\lambda}}(v, m)$ may not be reached: indeed, let $(x_k)_{k \geq k_0}$ be a sequence

of points in \mathbb{R}^n such that the balls $B^k = x_k + B_{(m/\alpha_n)^{1/n}}$ are pairwise disjoint and choose $k_0 \in \mathbb{N}$ and $v \in L^\infty(\mathbb{R}^n)$ so that $2^{-k_0} \leq \|v\|_\infty$, $v|_{B^k} = (\|v\|_\infty - 2^{-k}) e_r(\cdot - x_k)$ and $v = 0$ outside the balls B^k ; for each $\Omega \in \mathcal{C}$ with $|\Omega| \leq m$, one has $\lambda_1(\Omega, v|_\Omega) > \underline{\lambda}(v, m) = F_n(m, \|v\|_\infty)$.

Consider now other optimization problems where both Ω and v vary. Let $m > 0$ and $\tau \geq 0$ be fixed. As was already mentioned in Section 1, there holds $\inf_{\Omega \in \mathcal{C}, |\Omega| \leq m, \|v\|_\infty \leq \tau} \lambda_1(\Omega, v) = \inf_{\Omega \in \mathcal{C}, |\Omega| \leq m} \underline{\lambda}(\Omega, \tau) = \lambda_1(\Omega^*, \tau e_r)$. As for $\bar{\lambda}(v, m)$, the optimization problem for $\underline{\lambda}(\Omega, \tau)$ when $|\Omega| = m$ with a supremum instead of the infimum has the following solution:

$$\sup_{\Omega \in \mathcal{C}, |\Omega|=m} \underline{\lambda}(\Omega, \tau) = +\infty \quad (\text{whence} \quad \sup_{\Omega \in \mathcal{C}, |\Omega|=m} \bar{\lambda}(\Omega, \tau) = +\infty).$$

Finally, the characterization of $\inf_{\Omega \in \mathcal{C}, |\Omega|=m} \bar{\lambda}(\Omega, \tau)$ for given $m > 0$ and $\tau \geq 0$, is an open question.

4. Sketch of the proofs of Theorems 1.1 and 2.1

We now turn to the strategy of the proof of our results. Remember first that the proof of the classical Rayleigh–Faber–Krahn inequality (5) relies on two fundamental tools. The first one is a variational formulation of $\lambda_1(\Omega)$: $\lambda_1(\Omega) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \int_\Omega |\nabla u(x)|^2 dx / \int_\Omega u(x)^2 dx$, which relies heavily on the symmetry of the Laplacian. The second ingredient is the spherical Schwarz rearrangement, which makes this Rayleigh quotient decrease.

When $v \neq 0$, the operator $-\Delta + v \cdot \nabla$ is non-self-adjoint, and there is no simple variational formulation of its first eigenvalue such as above. Therefore, it seems impossible to adapt the ‘classical’ proof to prove Theorem 1.1. However, the proof of Theorem 1.1 also relies on a new type of rearrangement argument, which is definitely different from the usual rearrangement of functions.

First, using essentially the maximum principle and the Hopf lemma, one establishes Theorem 2.1. Once this is done, we are reduced to proving that $\underline{\lambda}(\Omega, \tau) \geq \underline{\lambda}(\Omega^*, \tau)$ for all $\tau \geq 0$, and that equality holds only when Ω is a ball.

To that purpose, we consider the positive function $\underline{\varphi}$ satisfying (7), and define a suitable spherical rearrangement of $\underline{\varphi}$, different from the Schwarz symmetrization. Namely, call R the radius of Ω^* ; define

$$\Omega_a = \{x \in \Omega, a < \underline{\varphi}(x) \leq 1\} \quad \text{for all } 0 \leq a < 1 \quad (\text{assuming up to multiplication that } \|\varphi\|_\infty = 1)$$

and define $\rho(a) \in (0, R]$ such that $|\Omega_a| = |B_{\rho(a)}|$. Define also $\rho(1) = 0$. The function $\rho : [0, 1] \rightarrow [0, R]$ is decreasing, continuous, one-to-one and onto. Then, the rearrangement of $\underline{\varphi}$ is the radially decreasing function $u : \Omega^* \rightarrow \mathbb{R}$ vanishing on $\partial\Omega^*$ such that, for all $0 \leq a < 1$,

$$\int_{\Omega_a} \Delta \underline{\varphi}(x) dx = \int_{B_{\rho(a)}} \Delta u(x) dx.$$

The fundamental inequality satisfied by u , which is also the key point in the proof of Theorem 1.1, is the fact that, for all $x \in \overline{\Omega^*}$,

$$u(x) \geq \rho^{-1}(|x|). \tag{9}$$

Strictly speaking, this fact is not completely correct, because the function $\underline{\varphi}$ is not regular enough, and we have to deal with suitable approximations of $\underline{\varphi}$ (see [8]). Let us just mention that the proof of (9) relies, among other things, on the usual isoperimetric inequality in \mathbb{R}^n . From (9) and arguments involving the maximum principle and the Hopf lemma again, we derive the conclusion of Theorem 1.1.

Finally, using again the same construction and the isoperimetric inequality, we prove that equality in Theorem 1.1 is attained if, and only if, up to translation, $\Omega = \Omega^*$ and $v = \tau e_r$.

Remark 1. Notice that the proof of Theorem 1.1 given in [8] still works for $\tau = 0$ and then provides an alternative proof of the Rayleigh–Faber–Krahn inequality (5) for the Dirichlet Laplacian.

All detailed proofs and further results are given in [8].

References

- [1] M.S. Ashbaugh, R.D. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, *Ann. Math.* 135 (1992) 601–628.
- [2] M.S. Ashbaugh, R.D. Benguria, Isoperimetric bounds for higher eigenvalue ratios for the n -dimensional fixed membrane problem, *Proc. Roy. Soc. Edinburgh Sect. A* 123 (1993) 977–985.
- [3] M.S. Ashbaugh, R.D. Benguria, On Rayleigh’s conjecture for the clamped plate and its generalization to three dimensions, *Duke Math. J.* 78 (1995) 1–17.
- [4] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, *Commun. Pure Appl. Math.* 47 (1994) 47–92.
- [5] D. Bucur, A. Henrot, Minimization of the third eigenvalue of the Dirichlet Laplacian, *Proc. Roy. Soc. London Ser. A* 456 (2000) 985–996.
- [6] S.-Y. Cheng, K. Oden, Isoperimetric inequalities and the gap between the first and second eigenvalues of an Euclidean domain, *J. Geom. Anal.* 7 (1997) 217–239.
- [7] G. Faber, Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, *Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang, 1923*, pp. 169–172.
- [8] F. Hamel, N. Nadirashvili, E. Russ, A Faber–Krahn inequality with drift, Preprint.
- [9] A. Henrot, Minimization problems for eigenvalues of the Laplacian, *J. Evol. Eq.* 3 (2003) 443–461.
- [10] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, *Math. Ann.* 94 (1925) 97–100.
- [11] E. Krahn, Über Minimaleigenschaft der Kugel in drei und mehr Dimensionen, *Acta Commun. Univ. Tartu (Dorpat)* A 9 (1926) 1–44.
- [12] P. Marcellini, Bounds for the third membrane eigenvalue, *J. Differential Equations* 37 (1980) 438–443.
- [13] N.S. Nadirashvili, Rayleigh’s conjecture on the principal frequency of the clamped plate, *Arch. Rational Mech. Anal.* 129 (1995) 1–10.
- [14] R. Osserman, The isoperimetric inequality, *Bull. Amer. Math. Soc.* 84 (1978) 1182–1238.
- [15] G. Pólya, On the characteristic frequencies of a symmetric membrane, *Math. Z.* 63 (1955) 331–337.
- [16] G. Pólya, On the eigenvalues of vibrating membranes, *Proc. London Math. Soc.* 11 (3) (1961) 100–105.
- [17] J.W.S. Rayleigh, *The Theory of Sound*, second ed. revised and enlarged (in 2 vols.), Dover, New York, 1945 (republication of the 1894/1896 edition).
- [18] G. Szegő, Inequalities for certain eigenvalues of a membrane of given area, *J. Rational Mech. Anal.* 3 (1954) 343–356.
- [19] H.F. Weinberger, An isoperimetric inequality for the N -dimensional free membrane problem, *J. Rational Mech. Anal.* 5 (1956) 633–636.