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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 281–286

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Partial Differential Equations/Probability Theory

Variational solutions for a class of fractional stochastic partial differential equations

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Received 18 October 2004; accepted 4 January 2005

Presented by Pierre-Louis Lions

Abstract

In this Note we present new results regarding the existence, the uniqueness and the equivalence of two notions of variational solution related to a class of non autonomous, semilinear, stochastic partial differential equations defined on an open bounded domain $D \subset \mathbb{R}^d$. The equations we consider are driven by an infinite-dimensional noise derived from an $L^2(D)$-valued fractional Wiener process W^{H} with Hurst parameter $\mathrm{H} \in (\frac{1}{\gamma+1}, 1)$, where $\gamma \in (0, 1]$ denotes the Hölder exponent of the derivative of the nonlinearity that appears in the stochastic term. ***To cite this article: D. Nualart, P.-A. Vuillermot, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Solutions variationnelles pour une classe d'équations aux dérivées partielles stochastiques fractionnaires. Dans cette Note nous présentons des résultats nouveaux concernant l'existence, l'unicité et l'équivalence de deux notions de solution variationnelle relatives à une classe d'équations aux dérivées partielles stochastiques semilinéaires non autonomes définies dans un ouvert borné $D \subset \mathbb{R}^d$. Les équations que nous considérons sont dirigées par un bruit en dimension infinie déduit d'un processus de Wiener fractionnaire W^{H} à valeurs dans $L^2(D)$ de paramètre de Hurst $\mathrm{H} \in (\frac{1}{\gamma+1}, 1)$, où $\gamma \in (0, 1]$ est l'exposant de Hölder de la dérivée de la nonlinéarité apparaissant dans le terme stochastique. ***Pour citer cet article : D. Nualart, P.-A. Vuillermot, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.01.006

Version française abrégée

La numérotation, la notation et la terminologie que nous utilisons ici ainsi que les hypothèses (C), (K), (H_γ) et (I) se réfèrent directement aux formules, à la notation, à la terminologie et aux hypothèses (C), (K), (H_γ) et (I) de la version principale en anglais. Nous nous intéressons à l'existence, à l'unicité et à l'équivalence de deux notions de solution variationnelle relatives au problème (2). Notre résultat principal est le suivant :

Théorème 0.1. *Supposons que* (C), (K), (H_γ) *et* (I) *soient satisfaites*; *alors le problème* (2) *possède une solution variationnelle* $u_{\mathrm{I},\varphi}$ *de type* I *et une solution variationnelle* $u_{\mathrm{II},\varphi}$ *de type* II *telles que* $u_{\mathrm{I},\varphi}(\cdot,t) = u_{\mathrm{II},\varphi}(\cdot,t)$ *p.p. dans* $L^2(D)$ *pour chaque* $t \in [0,T]$. *De plus, si la fonction* h *est affine,* $u_{\mathrm{I},\varphi}$ *est l'unique solution de type* I *de* (2) *et* $u_{\mathrm{II},\varphi}$ *en est l'unique solution de type* II.

Nous mettons ainsi en évidence l'existence, l'unicité et l'indiscernabilité des deux notions de solution variationnelle introduites ci-dessous. Notre démonstration de l'existence d'une solution variationnelle $u_{\mathrm{I},\varphi}$ repose sur l'existence et la convergence d'un schéma de Faedo–Galerkin convenable associé à (2). Nous pouvons ensuite démontrer que $u_{\mathrm{I},\varphi}$ est nécessairement une solution variationnelle de type II, ceci grâce à de nouvelles propriétés de continuité de l'intégrale stochastique que nous utilisons conjointement avec certaines propriétés d'approximation polynomiale des fonctions test admissibles. Nous obtenons l'unicité de $u_{\mathrm{I},\varphi}$ lorsque h est une fonction affine en utilisant le fait que g est lipschitzienne, puis nous pouvons finalement déduire les autres affirmations du théorème en observant que toute solution variationnelle de type II de (2) en est une solution variationnelle de type I. À notre connaissance, le théorème énoncé est nouveau : nous ne connaissons en effet aucun résultat de ce type pour les équations différentielles ou les équations intégrales stochastiques fractionnaires, bien que ces équations aient fait l'objet de plusieurs travaux récents (voir par exemple [6–10,12,14,15] et leurs références), et bien que plusieurs notions de solution variationnelle existent déjà depuis longtemps pour des problèmes tels que (2) lorsque le bruit est dérivé d'un processus de Wiener standard (voir par exemple [11,17,18]). Notre travail a été motivé entre autres choses par le fait qu'il existe nombre d'applications récentes s'appuyant sur une modélisation mathématique faisant intervenir soit des mouvements browniens fractionnaires, soit des équations différentielles ordinaires ou aux dérivées partielles dirigées par des bruits dérivant de tels mouvements browniens (voir par exemple [4,5,8–10,13] et leurs références). Nous renvoyons le lecteur à la version principale en anglais pour une description plus complète de nos résultats, dont nous donnons les démonstrations détaillées en [16].

1. Introduction, notation and main result

Many applications in engineering, the natural sciences and the world of finance, to name only a few, call for a mathematical modelling in terms of fractional Brownian motions or in terms of differential equations driven by a finite or an infinite-dimensional fractional noise (see, for instance, [4,5,8–10,13] and their references). In this Note we present new results regarding a class of non autonomous, semilinear, fractional stochastic partial differential equations that have potential applications to the above fields. In what follows we assume that all the functional spaces are real and use the standard notations for the usual spaces of Lebesgue integrable functions and their norms. For $d \in \mathbb{N}^+$ let $D \subset \mathbb{R}^d$ be open, bounded and assume that its boundary ∂D be sufficiently smooth. Let C be a linear, self-adjoint, positive, non degenerate trace-class operator in $L^2(D)$. In the following we write $(e_i)_{i\in\mathbb{N}^+}$ for an orthonormal basis of $L^2(D)$ consisting of eigenfunctions of the operator C and $(\lambda_i)_{i\in\mathbb{N}^+}$ for the sequence of the corresponding eigenvalues. Let $((B_i^{\mathrm{H}}(t))_{t\in\mathbb{R}_0^+})_{i\in\mathbb{N}^+}$ be a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter $\mathrm{H} \in (\frac{1}{2}, 1)$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and starting at the origin. We define the $L^2(D)$-valued, fractional Wiener process $(W^{\mathrm{H}}(\cdot,t))_{t\in\mathbb{R}_0^+}$ by

$$W^{\mathrm{H}}(\cdot,t) := \sum_{i=1}^{+\infty} \lambda_i^{1/2} e_i(\cdot) B_i^{\mathrm{H}}(t) \tag{1}$$

where the series (1) converges in the strong topology of $L^2(D)$ a.s. by virtue of the basic properties of the $B_i^{\mathrm{H}}(t)$'s and the fact that C is trace-class. Let $T \in \mathbb{R}^+$ and let us consider the class of real, parabolic, initial-boundary value problems given by

$$\begin{aligned}
&\mathrm{d}u(x,t) = \big(\mathrm{div}\big(k(x,t)\nabla u(x,t)\big) + g\big(u(x,t)\big)\big)\,\mathrm{d}t + h\big(u(x,t)\big)W^{\mathrm{H}}(x,\mathrm{d}t),\\
&(x,t) \in D \times (0,T],\\
&u(x,0) = \varphi(x), \quad x \in D,\\
&\frac{\partial u(x,t)}{\partial n(k)} = 0, \quad (x,t) \in \partial D \times (0,T].
\end{aligned} \tag{2}$$

In the preceding equations the function k is matrix-valued and the last relation stands for the conormal derivative of u relative to k; moreover, we assume that the following hypotheses hold:

(C) We have $e_i \in L^\infty(D)$ for every i and $\sum_{i=1}^{+\infty} \lambda_i^{1/2} \|e_i\|_\infty < +\infty$.

(K) The function $k: D \times (0,T] \mapsto \mathbb{R}^{d^2}$ is Lebesgue-measurable and satisfies the symmetry relation $k_{i,j}(\cdot) = k_{j,i}(\cdot)$ for every $i, j \in \{1,\ldots,d\}$. Moreover, there exist constants $\underline{k}, \bar{k} \in \mathbb{R}^+$ such that the inequalities

$$\underline{k}|q|^2 \leqslant \big(k(x,t)q, q\big)_{\mathbb{R}^d} \leqslant \bar{k}|q|^2$$

hold uniformly in $(x,t) \in D \times (0,T]$ for all $q \in \mathbb{R}^d$, where we have written $(\cdot,\cdot)_{\mathbb{R}^d}$ for the standard Euclidean inner product in $\mathbb{R}^d$.

(H_γ) The functions $g, h: \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz continuous; moreover, the derivative $h': \mathbb{R} \mapsto \mathbb{R}$ exists, is Hölder continuous with exponent $\gamma \in (0,1]$ and bounded; finally we have $\mathrm{H} \in (\frac{1}{\gamma+1}, 1)$.

(I) The initial condition φ is an $L^2(D)$-valued random variable.

Formally, relations (2) define a class of non autonomous, semilinear, stochastic initial-boundary value problems driven by an infinite-dimensional fractional noise. Our purpose in this Note is to introduce two notions of variational solution for (2) and to present new results regarding their existence, their uniqueness and their indistinguishability. To this end we write $(\cdot,\cdot)_2$ for the standard inner product in $L^2(D)$, $H^1(D)$ for the usual Sobolev space of real-valued functions on D and $\|\cdot\|_{1,2}$ for the corresponding norm. We fix once and for all an $\alpha \in (1-H, \frac{\gamma}{\gamma+1})$ and introduce the Banach space $\mathcal{B}^{\alpha,2}([0,T]; L^2(D))$ of all Lebesgue-measurable mappings $u: [0,T] \mapsto L^2(D)$ endowed with the norm

$$\|u\|_{\alpha,2}^2 := \Big(\operatorname*{esssup}_{t\in[0,T]} \big\|u(t)\big\|_2\Big)^2 + \int_0^T \mathrm{d}t \left(\int_0^t \mathrm{d}\tau \frac{\|u(t)-u(\tau)\|_2}{(t-\tau)^{\alpha+1}}\right)^2 < +\infty.$$

Our first notion of variational solution for (2) is one where the admissible test functions are independent of the time variable.

Definition 1.1. We say the $L^2(D)$-valued random field $(u_{\mathrm{I}}(\cdot,t))_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *variational solution of type* I to problem (2) if the following two conditions hold:

(1) We have $u_{\mathrm{I}} \in L^2(0,T; H^1(D)) \cap \mathcal{B}^{\alpha,2}([0,T]; L^2(D))$ a.s.

(2) The integral relation

$$\big(v, u_{\mathrm{I}}(\cdot,t)\big)_2 = (v,\varphi)_2 - \sum_{i,j=1}^{d} \int_0^t \mathrm{d}\tau \big(v_{x_i}, k_{i,j}(\cdot,\tau) u_{\mathrm{I},x_j}(\cdot,\tau)\big)_2 + \int_0^t \mathrm{d}\tau \big(v, g\big(u_{\mathrm{I}}(\cdot,\tau)\big)\big)_2 + \sum_{i=1}^{+\infty} \lambda_i^{1/2} \int_0^t \big(v, h\big(u_{\mathrm{I}}(\cdot,\tau)\big) e_i\big)_2 B_i^{\mathrm{H}}(\mathrm{d}\tau) \tag{3}$$

holds a.s. for every $v \in H^1(D)$ and every $t \in [0,T]$.

From the preceding definition and the above hypotheses, we easily infer that each term in (3) is finite a.s. In particular, we can define each one-dimensional stochastic integral in (3) as a generalized Stieltjes integral in the sense of [14,15,19] or [20].

It is well-known that various notions of variational solution already exist for problems of the form (2) when the noise is derived from a standard Wiener process (see, for instance, [11] and [17]); over the years these notions have played an important rôle in a number of situations, particularly in those dealing with the analysis of long-time behavior phenomena in population dynamics and population genetics (see, for instance, [1–3] and their references), aside from being equivalent to other notions of solution as was recently shown in [18]. On the other hand, we are not aware of any works concerning the existence of variational solutions for problems driven by a fractional noise such as (2), though there have been several recent articles devoted to the mathematical analysis of quite a few types of fractional stochastic differential and integral equations (see, for instance, [6–10,12,14,15] and their references).

In this context there is yet another natural notion of variational solution we can think of for (2), namely, one where the admissible test functions depend on the basic space–time variable (x,t). For every $t \in (0,T]$ let $H^1(D \times (0,t))$ be the Sobolev space of all real-valued functions $v \in L^2(D \times (0,t))$ that possess distributional derivatives $v_{x_i} \in L^2(D \times (0,t))$ for every $i \in \{1,\ldots,d\}$ along with a distributional time-derivative $v_\tau \in L^2(D \times (0,t))$, the norm of $H^1(D \times (0,t))$ being defined in the usual way.

Definition 1.2. We say the $L^2(D)$-valued random field $(u_{\mathrm{II}}(\cdot,t))_{t\in[0,T]}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ is a *variational solution of type* II to problem (2) if the first condition of Definition 1.1 holds, and if the integral relation

$$\big(v(\cdot,t), u_{\mathrm{II}}(\cdot,t)\big)_2 = \big(v(\cdot,0),\varphi\big)_2 + \int_0^t \mathrm{d}\tau \big(v_\tau(\cdot,\tau), u_{\mathrm{II}}(\cdot,\tau)\big)_2 - \sum_{i,j=1}^{d} \int_0^t \mathrm{d}\tau \big(v_{x_i}(\cdot,\tau), k_{i,j}(\cdot,\tau) u_{\mathrm{II},x_j}(\cdot,\tau)\big)_2 + \int_0^t \mathrm{d}\tau \big(v(\cdot,\tau), g\big(u_{\mathrm{II}}(\cdot,\tau)\big)\big)_2 + \sum_{i=1}^{+\infty} \lambda_i^{1/2} \int_0^t \big(v(\cdot,\tau), h\big(u_{\mathrm{II}}(\cdot,\tau)\big) e_i\big)_2 B_i^{\mathrm{H}}(\mathrm{d}\tau) \tag{4}$$

holds a.s. for every $v \in H^1(D \times (0,t))$ and every $t \in [0,T]$, where $x \mapsto v(x,0) \in L^2(D)$ and $x \mapsto v(x,t) \in L^2(D)$ denote the Sobolev traces of v on D and $D \times \{\tau \in \mathbb{R}\colon\ \tau = t\}$, respectively.

Again we can prove that every term in (4) is well-defined and finite a.s.

Our main result is the following:

Theorem 1.3. *Assume that Hypotheses* (C), (K), (H_γ) *and* (I) *hold. Then, problem* (2) *possesses a variational solution $u_{\mathrm{I},\varphi}$ of type* I *and a variational solution $u_{\mathrm{II},\varphi}$ of type* II, *such that $u_{\mathrm{I},\varphi}(\cdot,t) = u_{\mathrm{II},\varphi}(\cdot,t)$ a.s. in $L^2(D)$ for every $t \in [0,T]$. Moreover, if h is an affine function, $u_{\mathrm{I},\varphi}$ is the only variational solution of type* I *to* (2) *and $u_{\mathrm{II},\varphi}$ is its only variational solution of type* II.

2. Remarks and brief sketch of the proof

(1) Let $(w_n)_{n\in\mathbb{N}^+}$ be an orthonormal basis of $L^2(D)$ such that $(c_n w_n)_{n\in\mathbb{N}^+}$ be an orthonormal basis of $H^1(D)$ for some suitably chosen coefficients c_n (such a basis always exists by virtue of standard elliptic theory); for each $N \in \mathbb{N}^+$, let V_N be the finite-dimensional subspace of $L^2(D)$ generated by $\{w_1, \ldots, w_N\}$ and write φ_N for the orthogonal projection of the initial condition φ onto V_N. Our existence proof of a variational solution of type I rests upon the existence and the convergence of the Faedo–Galerkin scheme associated with (2) that consists of the approximation sequence $(u_N)_{N\in\mathbb{N}^+}$ where $u_N \in L^2(0,T;V_N) \cap \mathcal{B}^{\alpha,2}([0,T];V_N)$, whose elements satisfy the relation

$$\begin{aligned}\big(w_n, u_N(\cdot,t)\big)_2 = (w_n,\varphi_N)_2 &- \sum_{i,j=1}^{d} \int_0^t \mathrm{d}\tau \big(w_{n,x_i}, k_{i,j}(\cdot,\tau) u_{N,x_j}(\cdot,\tau)\big)_2 \\ &+ \int_0^t \mathrm{d}\tau \big(w_n, g\big(u_N(\cdot,\tau)\big)\big)_2 \\ &+ \sum_{i=1}^{+\infty} \lambda_i^{1/2} \int_0^t \big(w_n, h\big(u_N(\cdot,\tau)\big)e_i\big)_2 B_i^{\mathrm{H}}(\mathrm{d}\tau)\end{aligned}$$

a.s. for every $n \in \{1,\ldots,N\}$ and every $t \in [0,T]$. Our proof that such a sequence actually exists is built on the general theory of [15] for systems of fractional stochastic ordinary differential equations, while our proof of its convergence relies upon the availability of good a priori estimates that are uniform in N and upon a new compact embedding theorem for the Banach space $L^2(0,T;H^1(D)) \cap \mathcal{B}^{\alpha,2}([0,T];L^2(D))$.

(2) In order to prove that (2) possesses a variational solution of type II, we argue indirectly by showing that $u_{\mathrm{I},\varphi}$ itself is necessarily of type II; we obtain this result by proving new continuity properties of the stochastic integral in (4) with respect to the strong topology of $\mathcal{B}^{\alpha,2}([0,T];L^2(D))$, and by combining these continuity properties with suitable polynomial approximations of the admissible test functions that appear in (4). The uniqueness statement of the theorem when h is an affine function is then a consequence of the Lipschitz property of g, while the remaining statements follow from the fact that every variational solution of type II is trivially of type I. We do not know whether the uniqueness statement still holds for an arbitrary h satisfying Hypothesis (H_γ).

We refer the reader to [16] for more details and for the complete proofs of the above results.

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