Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 340 (2005) 305–308
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Probability Theory

Invariant measures of stochastic partial differential equations and conditioned diffusions

Maria G. Reznikoff [a], Eric Vanden-Eijnden [b]

[a] *Institute for Applied Mathematics, University of Bonn, Wegelerstraße 10, 53115 Bonn, Germany*
[b] *Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA*

Received 29 September 2004; accepted after revision 14 December 2004

Available online 2 February 2005

Presented by Paul Malliavin

Abstract

This work establishes and exploits a connection between the invariant measure of stochastic partial differential equations (SPDEs) and the law of bridge processes. Namely, it is shown that the invariant measure of $u_t = u_{xx} + f(u) + \sqrt{2\varepsilon}\,\eta(x,t)$, where $\eta(x,t)$ is a space–time white-noise, is identical to the law of the bridge process associated to $\mathrm{d}U = a(U)\,\mathrm{d}x + \sqrt{\varepsilon}\,\mathrm{d}W(x)$, provided that a and f are related by $\varepsilon a''(u) + 2a'(u)a(u) = -2f(u)$, $u \in \mathbb{R}$. Some consequences of this connection are investigated, including the existence and properties of the invariant measure for the SPDE on the line, $x \in \mathbb{R}$. ***To cite this article: M.G. Reznikoff, E. Vanden-Eijnden, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Mesures invariantes d'équations aux dérivées partielles stochastiques et diffusions conditionées. On montre et exploite une connection entre la mesure invariante d'équations aux dérivées partielles stochastiques et les lois de processus ponts. En l'occurence, on montre que la mesure invariante de $u_t = u_{xx} + f(u) + \sqrt{2\varepsilon}\,\eta(x,t)$, où $\eta(x,t)$ est un bruit blanc spatio-temporel, est la même que la loi du processus pont associé à $\mathrm{d}U = a(U)\,\mathrm{d}x + \sqrt{\varepsilon}\,\mathrm{d}W(x)$, pourvu que a et f soient reliés comme $\varepsilon a''(u) + 2a'(u)a(u) = -2f(u)$, $u \in \mathbb{R}$. Quelques conséquences de cette connection sont étudiées, comme l'existence et les propriétés d'une mesure invariante de l'équations aux dérivées partielle stochastique sur la ligne, $x \in \mathbb{R}$. ***Pour citer cet article : M.G. Reznikoff, E. Vanden-Eijnden, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail addresses: reznikoff@iam.uni-bonn.de (M.G. Reznikoff), eve2@cims.nyu.edu (E. Vanden-Eijnden).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.12.025

1. Introduction and main results

Consider the stochastic partial differential equation

$$u_t = u_{xx} + f(u) + \sqrt{2\varepsilon}\,\eta(x,t), \quad x \in [-L, L],\ t > 0, \tag{1}$$

where η is a space–time white noise, i.e. a Gaussian field with mean zero and covariance $\mathbb{E}(\eta(x,t)\eta(y,s)) = \delta(x-y)\delta(t-s)$ (formally). Suppose that

(A) $f(u) = -F'(u)$ with $F \in C^\infty(\mathbb{R})$, $F \geqslant 0$, and $F \to \infty$ as $|u| \to \infty$.

Then (1) can be interpreted by transformation into an integral equation using the solution of the linear part [1,3,4] and the invariant measure of (1) exists and is unique [1,3]. Since (1) is the L^2-gradient flow on $\int_{-L}^{L}(\frac{1}{2}u_x^2 + F(u))\,\mathrm{d}x$, plus perturbations, formally this invariant measure is

$$\exp\left(-\frac{1}{\varepsilon}\int_{-L}^{L}\left(\frac{1}{2}u_x^2 + F(u)\right)\mathrm{d}x\right)Du. \tag{2}$$

(2) can be interpreted [3] using the invariant measure of

$$v_t = v_{xx} + \sqrt{2\varepsilon}\,\eta(x,t) \tag{3}$$

as a reference measure, with density

$$\exp\left(-\frac{1}{\varepsilon}\int_{-L}^{L} F(v)\,\mathrm{d}x\right). \tag{4}$$

We propose a different method. The invariant measure of (3) coincides with the law of the Brownian bridge on $[-L, L]$ scaled by $\sqrt{\varepsilon}$. Building on this observation, we show that one can absorb the density (4) by choosing a more natural bridge process. More precisely, we have:

Theorem 1.1 (The invariant measure and the bridge measure). *Under assumption* (A)*, the invariant measure of* (1) *subject to $u(-L,t) = u_-$, $u(L,t) = u_+$ is* identical *to the law of the bridge process defined by*

$$\mathrm{d}U = a(U)\,\mathrm{d}x + \sqrt{\varepsilon}\,\mathrm{d}W \tag{5}$$

conditioned on $U(-L) = u_-$, $U(L) = u_+$, provided that a satisfies

$$\varepsilon a''(u) + 2a'(u)a(u) = -2f(u), \quad u \in \mathbb{R}. \tag{6}$$

The conditioning $U(-L) = u_-$, $U(L) = u_+$ defines a bridge process, and it can be realized by means of the h-transform [5], which amounts to replacing $a(u)$ in (5) by

$$\tilde{a}(u) = a(u) + \varepsilon\partial_u \log\bigl(p^u_{L-x}(u_+)\bigr),$$

where $p^u_x(u')$ is the transition probability density function of the unconditioned process. The law of the bridge process is the law of the SDE with drift $\tilde{a}$. The proof of Theorem 1.1 follows by application of the Girsanov formula to bridge processes.

Theorem 1.1 is easy to use in reverse. Given a, it is straightforward to calculate f using (6) and check whether it satisfies assumption (A). The SPDE (1) thus offers an explicit way to sample bridge processes associated with (5), which may be interesting in some cases. Also interesting but less obvious is the question of finding the drift a given f, since it allows analysis of the invariant measure of the SPDE (1) by studying the bridge process associated with the SDE (5). There are many solutions of (6) corresponding to different choices of boundary conditions which are only defined locally (they blow-up at finite u). These are unacceptable in Theorem 1.1. There are also many global solutions $a \in C^2(\mathbb{R})$ which lead to identical bridge processes after conditioning. There is, however, a unique solution which distinguishes itself:

Theorem 1.2. *Under assumption* (A)*, there exists a unique, global solution* $a_\star \in C^2(\mathbb{R})$ *of* (6) *which satisfies*

$$\int_{\mathbb{R}} \exp\left(\frac{2}{\varepsilon}\int_0^u a_\star(v)\,\mathrm{d}v\right)\mathrm{d}u < \infty. \tag{7}$$

Since (7) is necessary and sufficient for the SDE to have a unique equilibrium probability density, Theorem 1.2 asserts that $a_\star$ is the unique solution of (6) for which the SDE (5) has an equilibrium probability density. The theorem is proved by noting that (6) can be transformed into the Schrödinger equation

$$\varepsilon^2 w'' - 2F(u)w = \lambda w, \quad \text{for } w(u) := \exp\left(\frac{1}{\varepsilon}\int_0^u a(v)\,\mathrm{d}v\right) \tag{8}$$

and $\lambda = \varepsilon a'(0) + a^2(0)$. By assumption (A), the only strictly positive, L^2-solution of this Schrödinger equation is its ground state [6]. By the definition of w in (8), (7) is the expression that the ground state belongs to $L^2(\mathbb{R})$.

From the existence and uniqueness of the equilibrium probability density of (5) with $a = a_\star$, it follows that the equilibrium process, $\{U(x)\}_{x\in\mathbb{R}}$, associated with (5) for $a = a_\star$ is well-defined, and its law is unique. By equivalence with the invariant measure of (1), this leads to:

Corollary 1.3. *The law of the equilibrium process,* $\{U(x)\}_{x\in\mathbb{R}}$*, associated with* (5) *for* $a = a_\star$ *is an invariant measure of*

$$u_t = u_{xx} + f(u) + \sqrt{2\varepsilon}\,\eta(x,t), \quad x \in \mathbb{R}. \tag{9}$$

2. Applications and generalizations

1. The density (4) suggests that as $\varepsilon \to 0$, the invariant measure is supported on functions which are concentrated around the minima, m_i, of F. Theorem 1.1 confirms this picture, at least when L is fixed. Consider for instance the situation when $f = u - u^3$, corresponding to $F = \frac{1}{4}(1-u^2)^2$. In this case,

$$a_\star(u) \to \operatorname{sgn}(u)\sqrt{F(u)} = \frac{\operatorname{sgn}(u)(1-u^2)}{\sqrt{2}} \quad \text{as } \varepsilon \to 0. \tag{10}$$

The limit is simply the patching together the two outer (or naive) solutions of $\varepsilon a' + a^2 = 2F$ (obtained by integrating (6) once). From (10) and standard results from the Wentzell–Freidlin theory of large deviations for SDEs [2], it follows that if L is kept fixed as $\varepsilon \to 0$, the solutions of (1) concentrate on the minimizers of the action

$$\int_{-L}^{L} \left| U' - \frac{\operatorname{sgn}(U)(1-U^2)}{\sqrt{2}} \right|^2 \mathrm{d}x, \tag{11}$$

subject to $U(-L) = u_-$, $U(L) = u_+$. If $u_- = u_+ = 1$ (resp. -1), the minimizer of (11) is indeed $U(x) = 1$ (resp. -1).

On the other hand, if $L \to \infty$ as $\varepsilon \to 0$ on a sequence such that $\varepsilon \log L \to C > \sqrt{2}/3$, another regime emerges. The functions on which the invariant measure is supported are no longer concentrated near a single minimum of F, and transitions between minima become almost certain. These transitions correspond to hopping over the $\sqrt{2}/3$ high barrier in the cusped potential $A := \operatorname{sgn}(u)(u^3/3 - u)/\sqrt{2}$ associated with a (i.e. $a = -A'$). In particular, the functions within the support of the invariant measure of (1) tend to piecewise constant functions taking the values ± 1, and the lengths of the successive domains are i.i.d. Poisson variables with mean

$$\ell = \left(\sqrt{\varepsilon\pi}\, 2^{-3/4} + \mathrm{o}(\sqrt{\varepsilon})\right) \mathrm{e}^{\sqrt{2}/(3\varepsilon)}. \tag{12}$$

2. One expects that Theorem 1.1 is valid for a class of SPDE wider than (1), obtained by changing either the boundary conditions or the metric over which the gradient flow is constructed, provided that the conditioning of the solutions of (5) is changed accordingly. For instance, suppose that one considers (1) with periodic boundary conditions on $[-L, L]$. Then the invariant measure of the SPDE is identical to the law of (5) on $\mathbb{R}$ conditioned on the subset of solutions which are $2L$-periodic.

Similarly, if instead of (1), one considers the H^{-1}-gradient flow on $\int_{-L}^{L} (\frac{1}{2} u_x^2 + F(u))\, \mathrm{d}x$ plus perturbations,

$$u_t = -\partial_{xx}^2 \left(u_{xx} + f(u)\right) + \sqrt{2\varepsilon}\, \partial_x \eta(x, t), \tag{13}$$

then formally the invariant measure is again (2) with the additional constraint that

$$\int_{-L}^{L} u(x, t)\, \mathrm{d}x = cst. \tag{14}$$

Correspondingly, one expects the invariant measure of (13) to be identical to the law of (5) conditioned on the solutions of the SDE satisfying $\int_{-L}^{L} U(x)\, \mathrm{d}x = cst$ together with the boundary conditions.

3. The results can be generalized to situations in which u is a vector-valued field, $u : [-L, L] \times [0, \infty) \to \mathbb{R}^n$, satisfying

$$u_t = u_{xx} - \nabla_u F(u) + \sqrt{2\varepsilon}\, \eta(x, t), \tag{15}$$

where η is a vector-valued space–time white-noise and F is some smooth potential bounded from below and growing at infinity. Then the invariant measure of (15) is identical to the law of the bridge process associated with

$$\mathrm{d}U = a(U)\, \mathrm{d}x + \sqrt{\varepsilon}\, \mathrm{d}W(x), \tag{16}$$

provided that $a = \varepsilon \nabla w / w$, where w is the ground state of the Schrödinger equation

$$\varepsilon^2 \Delta w - 2F(u) w = \lambda w. \tag{17}$$

Acknowledgements

We thank Gérard Ben Arous, Percy Deift, Weinan E, Bob Kohn, Henry McKean, Chuck Newman, and Esteban Tabak for interesting discussions. This work was partially supported by NSF via grants DMS01-01439, DMS02-09959 and DMS02-39625.

References

[1] W.G. Faris, G. Jona-Lasinio, Large fluctuations for a nonlinear heat equation with noise, J. Phys. A 15 (1982) 3025–3055.
[2] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, second ed., Springer-Verlag, New York, 1998.
[3] R. Marcus, Monotone parabolic Itô equations, J. Funct. Anal. 29 (1978) 275–286.
[4] E. Pardoux, Stochastic partial differential equations, a review, Bull. Sci. Math. 117 (1993) 29–47.
[5] R.G. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge University Press, Cambridge, 1995.
[6] M. Reed, B. Simon, Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, New York, 1972.