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Probability Theory

Existence of weak solutions to stochastic evolution inclusions

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Abstract

We prove the existence of a weak mild solution (or mild solution-measure) to the Cauchy problem for the semilinear stochastic differential inclusion in a Hilbert space $dX_t \in AX_t dt + F(t, X_t) dt + G(t, X_t) dW_t$ where W is a cylindrical Wiener process, A is a linear operator which generates a C_0 -semigroup, F and G are multifunctions with convex compact values satisfying a linear growth condition and a condition weaker than the Lipschitz condition. The weak solution is constructed in the sense of Young measures. In the case when F and G are single-valued, we obtain the existence of a strong solution. **To cite this article:** *A. Jakubowski et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

Existence de solutions faibles d'inclusions d'évolution stochastiques. Nous démontrons l'existence d'une solution d'évolution faible (ou solution-mesure d'évolution) de l'inclusion différentielle stochastique dans un espace de Hilbert $dX_t \in AX_t dt + F(t, X_t) dt + G(t, X_t) dW_t$ où W est un mouvement brownien cylindrique, A est un opérateur linéaire qui engendre un semi-groupe de classe C_0 , F et G sont des multifonctions à valeurs convexes compactes vérifiant une condition de croissance linéaire ainsi qu'une condition plus générale que la condition de Lipschitz. La solution faible est construite au sens des mesures de Young. Lorsque F et G sont univoques, on obtient l'existence d'une solution forte. **Pour citer cet article :** *A. Jakubowski et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Dans ce qui suit, $0 < T < +\infty$ est un horizon fixé et $\mathfrak{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ est une base stochastique vérifiant les conditions habituelles. On se donne deux espaces de Hilbert séparables \mathbb{H} et \mathbb{U} et un (\mathcal{F}_t) -mouvement

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brownien W (éventuellement cylindrique) sur \mathbb{U} . Soit \mathbb{L} l'espace des opérateurs de Hilbert–Schmidt de \mathbb{U} dans \mathbb{H} . Dans ce travail, nous montrons l'existence d'une solution-mesure d'évolution au problème de Cauchy (1) ci-dessous, où X est à valeurs dans \mathbb{H} , A est un opérateur linéaire sur \mathbb{H} et F et G sont des applications mesurables à valeurs convexes compactes, respectivement dans \mathbb{H} et dans \mathbb{L} .

Une solution d'évolution forte de (1) a été obtenue par Da Prato et Frankowska [5] dans le cas où F et G vérifient une condition de Lipschitz par rapport à la deuxième variable. Nous affaiblissons nettement cette condition. En revanche, nous supposons les applications F et G déterministes et à valeurs convexes compactes, alors que dans [5] elles sont aléatoires et à valeurs fermées non nécessairement bornées.

On se donne une fois pour toutes un nombre $p > 2$. Si \mathbb{E} est un espace de Banach, on note $\mathfrak{K}_c(\mathbb{E})$ l'ensemble des compacts convexes non vides de \mathbb{E} , et $\text{Hausd}_{\mathbb{E}}$ la distance de Hausdorff sur $\mathfrak{K}_c(\mathbb{E})$. On se donne également les hypothèses suivantes :

- (HS) A engendre un semigroupe $(S(t))_{t \geq 0}$ de classe \mathcal{C}_0 .
- (HFG) $F : [0, T] \times \mathbb{H} \rightarrow \mathfrak{K}_c(\mathbb{H})$ et $G : [0, T] \times \mathbb{H} \rightarrow \mathfrak{K}_c(\mathbb{L})$ sont des applications mesurables telles que
 - (i) il existe une constante $C_{\text{growth}} > 0$ telle que, pour tout $(t, x) \in [0, T] \times \mathbb{H}$,

$$\text{Hausd}_{\mathbb{H}}(0, F(t, x)) \leq C_{\text{growth}}(1 + \|x\|), \quad \text{Hausd}_{\mathbb{L}}(0, G(t, x)) \leq C_{\text{growth}}(1 + \|x\|),$$

- (ii) pour tout $(t, x, y) \in [0, T] \times \mathbb{H} \times \mathbb{H}$,

$$(\text{Hausd}_{\mathbb{H}}(F(t, x), F(t, y)))^p \leq L(t, \|x - y\|^p),$$

$$(\text{Hausd}_{\mathbb{L}}(G(t, x), G(t, y)))^p \leq L(t, \|x - y\|^p),$$

où $L : [0, T] \times [0, +\infty] \rightarrow [0, +\infty]$ est une fonction continue donnée telle que pour tout $t \in [0, T]$, l'application $L(t, \cdot)$ soit croissante et concave et que, pour toute application mesurable $z : [0, T] \rightarrow [0, +\infty]$ et pour toute constante $K > 0$, on ait

$$\left[(\forall t \in [0, T]) z(t) \leq K \int_0^t L(s, z(s)) ds \right] \Rightarrow z = 0.$$

En particulier, pour tout $t \in [0, T]$, les multifonctions $F(t, \cdot)$ et $G(t, \cdot)$ sont continues. Ce type de fonction L est considéré notamment dans [11,1,12,3,4].

- (HI) $\xi \in L^p(\Omega, \mathcal{F}_0, \mathbb{P} |_{\mathcal{F}_0}; \mathbb{H})$.

Pour tout $t \in [0, T]$, $\mathcal{N}_c^p(\mathfrak{F}, [0, t]; \mathbb{H})$ désigne l'espace des processus continus (\mathcal{F}_t) -adaptés X à valeurs dans \mathbb{H} tels que $\mathbf{E}(\sup_{0 \leq s \leq t} \|X(s)\|_{\mathbb{H}}^p) < +\infty$.

Définition 0.1. On dit qu'un processus $X \in \mathcal{N}_c^p(\mathfrak{F}, [0, T]; \mathbb{H})$ est une *solution d'évolution (forte)* de (1) s'il existe des processus prévisibles f et g définis sur \mathfrak{F} vérifiant (5). On dit qu'un processus X est une *solution d'évolution faible* ou *solution-mesure d'évolution* de (1) s'il existe une base stochastique $\underline{\mathfrak{F}} = (\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ telle que

- (i) $\underline{\Omega}$ est de la forme $\underline{\Omega} = \Omega \times \Omega'$, $\underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}'$ pour une tribu \mathcal{F}' sur Ω' , $\underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}'_t$ pour une filtration continue à droite (\mathcal{F}'_t) sur (Ω', \mathcal{F}') et la probabilité μ vérifie $\mu(A \times \Omega') = \mathbb{P}(A)$ pour tout $A \in \mathcal{F}$,
- (ii) le processus W est un mouvement brownien sur $\underline{\mathfrak{F}}$ (on identifie ici toute variable aléatoire Y définie sur Ω avec la variable aléatoire $(\omega, \omega') \mapsto Y(\omega)$ définie sur $\underline{\Omega}$),
- (iii) $X \in \mathcal{N}_c^p(\underline{\mathfrak{F}}, [0, T]; \mathbb{H})$ et il existe des processus prévisibles f et g définis sur $\underline{\mathfrak{F}}$ vérifiant (5).

Les qualificatifs « fort » et « faible » sont donc à prendre au sens probabiliste. La terminologie *solution-mesure* est celle de Jacod et Mémin [7]. Si \mathcal{F}' est la tribu borélienne d'une topologie sur Ω' , on peut voir une solution-

mesure comme une mesure de Young. C’est le point de vue adopté par Pellaumail [10], qui a réinventé les mesures de Young sous le nom de *règles*.

Théorème 0.2. *Sous les hypothèses (HS), (HFG) et (HI), l’inclusion (1) admet une solution d’évolution faible.*

Dans le cas où \mathbb{H} et \mathbb{U} sont de dimension finie, une adaptation facile de la preuve de ce théorème et l’utilisation du point de Steiner montrent que (1) admet une solution d’évolution forte.

1. Formulation of the problem, statement of the result

1.1. Introduction

Throughout, $0 < T < +\infty$ is a fixed time and $\mathfrak{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is a stochastic basis satisfying the usual conditions. We are given two separable Hilbert spaces \mathbb{H} and \mathbb{U} and a (possibly cylindrical) $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion W on \mathbb{U} . We denote by \mathbb{L} the space of Hilbert–Schmidt operators from \mathbb{U} to \mathbb{H} . We prove the existence of a ‘weak’ (in the sense of probability) mild solution X to the Cauchy problem

$$\begin{cases} dX_t \in AX_t + F(t, X_t) dt + G(t, X_t) dW(t), \\ X(0) = \xi, \end{cases} \tag{1}$$

where X takes its values in \mathbb{H} , A is a linear operator on \mathbb{H} and F and G are measurable multivalued mappings, with convex compact values in \mathbb{H} and \mathbb{L} respectively.

The existence of a strong mild solution to (1) has been proved by Da Prato and Frankowska in [5] under a Lipschitz assumption. We replace this assumption by a much more general one. On the other hand, we assume that F and G have compact convex values (whereas they are only assumed to have closed values in [5]) and that they are deterministic (whereas they are random in [5]).

1.2. Notations, hypothesis, definitions

We are given a fixed number $p > 2$. For any $t \in [0, T]$, we denote by $\mathcal{N}_c^p(\mathfrak{F}, [0, t]; \mathbb{H})$ the space of (\mathcal{F}_t) -adapted \mathbb{H} -valued continuous processes X such that

$$\|X\|_{\mathcal{N}_c^p(\mathfrak{F}, [0, t]; \mathbb{H})}^p := \mathbf{E} \left(\sup_{0 \leq s \leq t} \|X(s)\|_{\mathbb{H}}^p \right) < +\infty.$$

If \mathbb{E} is a Banach space, we denote by $\mathfrak{K}_c(\mathbb{E})$ the set of nonempty convex compact subsets of \mathbb{E} , and by $\text{Hausd}_{\mathbb{E}}$ the Hausdorff distance on $\mathfrak{K}_c(\mathbb{E})$. We shall assume the following hypothesis:

(HS) A is the generator of a C_0 semigroup $(S(t))_{t \geq 0}$. In particular there exist $M > 0$ and $\beta \in]-\infty, +\infty[$ such that, for every $t \geq 0$,

$$\|S(t)\| \leq M e^{\beta t}. \tag{2}$$

For $t \in [0, T]$, we denote $M_t = \sup_{0 \leq s \leq t} M e^{\beta s}$.

(HFG) $F : [0, T] \times \mathbb{H} \rightarrow \mathfrak{K}_c(\mathbb{H})$ and $G : [0, T] \times \mathbb{H} \rightarrow \mathfrak{K}_c(\mathbb{L})$ are measurable mappings which satisfy:

(i) There exists a constant $C_{\text{growth}} > 0$ such that, for all $(t, x) \in [0, T] \times \mathbb{H}$,

$$\text{Hausd}_{\mathbb{H}}(0, F(t, x)) \leq C_{\text{growth}}(1 + \|x\|), \quad \text{Hausd}_{\mathbb{L}}(0, G(t, x)) \leq C_{\text{growth}}(1 + \|x\|).$$

(ii) For all $(t, x, y) \in [0, T] \times \mathbb{H} \times \mathbb{H}$,

$$(\text{Hausd}_{\mathbb{H}}(F(t, x), F(t, y)))^p \leq L(t, \|x - y\|^p),$$

$$(\text{Hausd}_{\mathbb{L}}(G(t, x), G(t, y)))^p \leq L(t, \|x - y\|^p),$$

where $L : [0, T] \times [0, +\infty] \rightarrow [0, +\infty]$ is a given continuous mapping such that

- (a) for every $t \in [0, T]$, the mapping $L(t, \cdot)$ is nondecreasing and concave,
- (b) for every measurable mapping $z : [0, T] \rightarrow [0, +\infty]$ and for every constant $K > 0$, the following implication holds true:

$$\left[(\forall t \in [0, T]) z(t) \leq K \int_0^t L(s, z(s)) ds \right] \Rightarrow z = 0. \tag{3}$$

In particular, for all $t \in [0, T]$, we have $L(t, 0) = 0$ for all $t \in [0, T]$, thus, by hypothesis (HFG)(ii), $F(t, \cdot)$ and $G(t, \cdot)$ are continuous for the Hausdorff distance. Such a function L is considered in e.g. [11,1,12,3,4]. Concrete examples can be found in [12].

(HI) $\xi \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}|_{\mathcal{F}_0}; \mathbb{H})$.

Recall (see [6]) that, under Hypothesis (HS), there exists a constant C_{Conv} such that, for any predictable process $Z \in L^p(\Omega \times [0, T]; \mathbb{L})$, we have

$$\mathbf{E} \left[\sup_{s \leq t} \left\| \int_0^s S(s-r)Z(r) dW(r) \right\|^p \right] \leq C_{\text{Conv}} t^{(p/2)-1} \mathbf{E} \int_0^t \|Z(s)\|_{\mathbb{L}}^p ds. \tag{4}$$

Definition 1.1. We say that a process $X \in \mathcal{N}_c^p(\mathfrak{F}, [0, T]; \mathbb{H})$ is a (*strong*) *mild solution* to (1) if there exist two predictable processes f and g defined on \mathfrak{F} such that

$$\begin{cases} X(t) = S(t)\xi + \int_0^t S(t-s)f(s) ds + \int_0^t S(t-s)g(s) dW(s), \\ f(s) \in F(s, X(s)) \text{ P-a.e.}, \quad g(s) \in G(s, X(s)) \text{ P-a.e.} \end{cases} \tag{5}$$

We say that a process X is a *weak mild solution* or a *mild solution-measure* to (1) if there exists a stochastic basis $\underline{\mathfrak{F}} = (\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ satisfying the following conditions:

- (i) $\underline{\Omega}$ has the form $\underline{\Omega} = \Omega \times \Omega'$, $\underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}'$ for some σ -algebra \mathcal{F}' on Ω' , $\underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}'_t$ for some right continuous filtration (\mathcal{F}'_t) on (Ω', \mathcal{F}') , and the probability μ satisfies $\mu(A \times \Omega') = \mathbb{P}(A)$ for every $A \in \mathcal{F}$.
- (ii) The process W is a Brownian motion on $\underline{\mathfrak{F}}$ (we identify here every random variable Y on Ω with the random variable $(\omega, \omega') \mapsto Y(\omega)$ defined on $\underline{\Omega}$).
- (iii) $X \in \mathcal{N}_c^p(\underline{\mathfrak{F}}, [0, T]; \mathbb{H})$ and there exist two predictable processes f and g defined on $\underline{\mathfrak{F}}$ satisfying (5).

So, the qualifiers ‘strong’ and ‘weak’ are to be taken in the probabilistic sense. The terminology *solution-measure* is that of [7]. If \mathcal{F}' is the Borel σ -algebra of some topology on Ω' , a solution-measure can also be seen as a Young measure. This is the point of view adopted by Pellaumail [10], who reinvented Young measures under the name of *rules*.

1.3. Statement of the results

Theorem 1.2. Under Hypothesis (HS), (HFG) and (HI), inclusion (1) has a weak mild solution.

An adaptation of our reasoning also yields a well-known strong existence result [3,4]: if F and G are single-valued, then (1) has a strong mild solution. If \mathbb{H} and \mathbb{U} are finite dimensional, considering the Steiner point of F and G , we deduce from this result that (1) has a strong mild solution.

2. Sketch of the proof

Let us denote by Φ the mapping which, with every continuous adapted \mathbb{H} -valued process X such that $\mathbf{E} \int_0^T \|X(s)\|^p ds < +\infty$, associates the set of all processes of the form $S(t)\xi + \int_0^t S(t-s)f(s) ds + \int_0^t S(t-s)g(s) dW(s)$, where f and g are predictable selections of $(\omega, t) \mapsto F(t, X(\omega, t))$ and $(\omega, t) \mapsto G(t, X(\omega, t))$ respectively. Let $C([0, T]; \mathbb{H})$ denote the space of continuous mappings from $[0, T]$ to \mathbb{H} . The proof of Theorem 1.2 relies on the following lemmas:

Lemma 2.1. *Let Λ be a set of continuous (\mathcal{F}_t) -adapted processes on \mathbb{H} . Assume that each element of Λ is in $L^p(\Omega \times [0, T]; \mathbb{H})$ and that Λ , considered as a set of $C([0, T]; \mathbb{H})$ -valued random variables, is tight. Assume furthermore Hypothesis (HS) and (HFG)(i). Then $\Phi(\Lambda)$ is a tight set of $C([0, T]; \mathbb{H})$ -valued random variables.*

If (\mathbb{E}, d) is a metric space and $\Lambda \subset \mathbb{E}$, we say that a subset Λ' of \mathbb{E} is an ϵ -net of Λ if $\inf_{X \in \Lambda} d(X, \Lambda') \leq \epsilon$ (note that Λ' is not necessarily a subset of Λ). Let Λ be a subset of $\mathcal{N}_c^p(\mathfrak{F}, [0, T]; \mathbb{H})$. For every $s \in [0, T]$, let $\Lambda_s \subset \mathcal{N}_c^p(\mathfrak{F}, [0, s]; \mathbb{H})$ be the set of restrictions to $[0, s]$ of elements of Λ . We denote

$$\Psi(\Lambda)(s) := \inf\{\epsilon > 0; \Lambda_s \text{ has a tight } \epsilon\text{-net in } \mathcal{N}_c^p(\mathfrak{F}, [0, s]; \mathbb{H})\}.$$

So, Λ is tight if and only if $\Psi(\Lambda)(T) = 0$. We denote $\Psi(\Lambda) := (\Psi(\Lambda)(s))_{0 \leq s \leq T}$. The family $\Psi(\Lambda)$ is called the *measure of noncompactness* of Λ (see [1,8] about measures of noncompactness).

Lemma 2.2. *Let Λ be a bounded subset of $\mathcal{N}_c^p(\mathfrak{F}, [0, T]; \mathbb{H})$. Assume Hypothesis (HFG). We then have $\Psi^p(\Phi \circ \Lambda)(t) \leq k \int_0^t L(s, \Psi^p(\Lambda)(s)) ds$ for some constant k which depends only on T, p, M_T , and C_{Conv} .*

Our proof of Lemma 2.2 draws inspiration from [1, Lemma 4.2.6] and uses Lemma 2.1.

Proof of Theorem 1.2: First Part. We build a tight sequence of approximating solutions through Tonelli’s scheme: For each $n \geq 1$, we define a process \tilde{X}_n on $[-1, T]$ by $\tilde{X}_n(t) = 0$ if $t \leq 0$ and, for $t \geq 0$,

$$\tilde{X}_n(t) = S(t)\xi + \int_0^t S\left(t - \left(s - \frac{1}{n}\right)\right) f_n(s) ds + \int_0^t S\left(t - \left(s - \frac{1}{n}\right)\right) g_n(s) dW(s),$$

where $f_n : \Omega \times [0, T] \rightarrow \mathbb{H}$ and $g_n : \Omega \times [0, T] \rightarrow \mathbb{L}$ are predictable and $f_n(s) \in F(s, \tilde{X}_n(s - 1/n))$ P-a.e. and $g_n(s) \in G(s, \tilde{X}_n(s - 1/n))$ P-a.e. We then set $X_n(t) = \xi$ for $t \leq 1/n$ and, for $t \in [1/n, T]$, $X_n(t) = \tilde{X}_n(t - 1/n) = S(t - 1/n)\xi + \int_0^{t-1/n} S(t-s)f_n(s) ds + \int_0^{t-1/n} S(t-s)g_n(s) dW(s)$. A calculation using (2), (4) and Gronwall Lemma shows that (X_n) is bounded in $L^p(\Omega \times [0, T]; \mathbb{H})$. Then, with the help of Lemma 2.2, we obtain

$$\Psi^p\left(\bigcup_n \{X_n\}\right)(t) \leq M_T^p \Psi^p\left(\bigcup_n \Phi(X_n)\right)(t) \leq M_T^p k \int_0^t L\left(s, \Psi^p\left(\bigcup_n \{X_n\}\right)(s)\right) ds.$$

From (3), we conclude that (X_n) is a tight set of $C([0, T]; \mathbb{H})$ -valued random variables. \square

Remark 1. If F and G are single-valued, replacing tight nets by finite nets in the definition of Ψ , an easy adaptation of all previous arguments shows that the sequence (X_n) provided by the Tonelli scheme is relatively compact in $\mathcal{N}_c^p(\mathfrak{F}, [0, T]; \mathbb{H})$. Moreover, the limit of any convergent subsequence of (X_n) is a strong mild solution of (1). Thus, in this case, (1) has a strong mild solution, as was proved in [3,4].

Proof of Theorem 1.2: Second Part. We now construct a weak mild solution to (1).

By Prohorov's compactness criterion for Young measures, we can extract a subsequence of (X_n) which converges stably (i.e. in the sense of Young measures) to a Young measure $\mu \in \mathcal{Y}(\Omega, \mathcal{F}, \mathbb{P}; C([0, T]; \mathbb{H}))$. For simplicity, we denote this extracted sequence by (X_n) .

Let \mathcal{C} be the Borel σ -algebra of $C([0, T]; \mathbb{H})$ and, for each $t \in [0, T]$, let \mathcal{C}_t be the sub- σ -algebra of \mathcal{C} generated by $C([0, t]; \mathbb{H})$. We define a stochastic basis $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ by $\underline{\Omega} = \Omega \times C([0, T]; \mathbb{H})$, $\underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{C}$, $\underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{C}_t$ and we define X_∞ on $\underline{\Omega}$ by $X_\infty(\omega, u) = u$. Clearly, X_∞ is $(\underline{\mathcal{F}}_t)$ -adapted. The random variables X_n can be seen as random elements defined on $\underline{\Omega}$, using the notation $X_n(\omega, u) := X_n(\omega)$ ($n \in \mathbb{N}$). Furthermore, X_n is $(\underline{\mathcal{F}}_t)$ -adapted for each n , and W is $(\underline{\mathcal{F}}_t)$ -adapted. By a result of Balder (e.g. [2]), each subsequence of (X_n) contains a further subsequence (X'_n) such that, for each subsequence (X''_n) of (X'_n) , we have $\lim_n \frac{1}{n} \sum_{i=1}^n \delta_{X''_i}(\omega) = \mu_\omega$ a.e. where δ_x is the probability concentrated on x and $\omega \mapsto \mu_\omega$ is the disintegration of μ . This entails that, for every $A \in \mathcal{C}_t$, the mapping $\omega \mapsto \mu_\omega(A)$ is \mathcal{F}_t -measurable. Then is easy to prove that W is an $(\underline{\mathcal{F}}_t)$ -Wiener process under the probability μ .

To prove that $X = X_\infty$ satisfies (5), we choose particular selections f_n and g_n . From a result of Kucia [9], we can find measurable selections \mathfrak{f} and \mathfrak{g} of F and G respectively such that $\mathfrak{f}(t, \cdot)$ and $\mathfrak{g}(t, \cdot)$ are continuous for every $t \in [0, T]$. Denoting $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, we set, for each $n \in \widehat{\mathbb{N}}$ and each $t \in [0, T]$,

$$f_n(t) = \mathfrak{f}(t, X_n(t)), \quad g_n(t) = \mathfrak{g}(t, X_n(t)),$$

$$Z_n(t) = -X_n(t) + S(t)\xi + \int_0^{t-1/n} S(t-s)f_n(s) ds + \int_0^{t-1/n} S(t-s)g_n(s) dW(s)$$

(with $1/\infty := 0$). The sequence (X_n, W, f_n, g_n) converges in law to $(X_\infty, W, f_\infty, g_\infty)$ in $C([0, T]; \mathbb{H}) \times C([0, T]; \mathbb{U}) \times L^p([0, T]; \mathbb{H}) \times L^p([0, T]; \mathbb{L})$. Using the fact that $p > 2$, we prove that, for every $t \in [0, T]$, the sequence $(Z_n(t))$ converges in law to $Z_\infty(t)$. But we deduce from (HFG)(i) and the boundedness of (X_n) in $L^p(\Omega \times [0, T]; \mathbb{H})$ that $Z_n(t)$ converges to 0 in probability. Thus, for every $t \in [0, T]$, we have $Z_\infty(t) = 0$ a.e. As Z_∞ is continuous, this means that $Z_\infty = 0$ a.e. Thus X is a weak mild solution to (1). \square

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