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Combinatorics

Rank, term rank and chromatic number of a graph

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Abstract

Let G be a graph with a nonempty edge set, we denote the rank of the adjacency matrix of G and the term rank of G, by $\mathrm{rk}(G)$ and $\mathrm{Rk}(G)$, respectively. It was conjectured [C. van Nuffelen, Amer. Math. Monthly 83 (1976) 265–266], for any graph G, $\chi(G) \leqslant \mathrm{rk}(G)$. The first counterexample to this conjecture was obtained by Alon and Seymour [J. Graph Theor. 13 (1989) 523–525]. Recently, Fishkind and Kotlov [Discrete Math. 250 (2002) 253–257] have proved that for any graph G, $\chi(G) \leqslant \mathrm{Rk}(G)$. In this Note we improve Fishkind–Kotlov upper bound and show that $\chi(G) \leqslant \frac{\mathrm{rk}(G)+\mathrm{Rk}(G)}{2}$. ***To cite this article: S. Akbari, H.-R. Fanaï, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Rang, rang maximal et nombre chromatique d'un graphe. Soit G un graphe avec un ensemble d'arête non vide. On note $\mathrm{rk}(G)$ le rang (réel) d'une matrice d'adjacence A de G et $\mathrm{Rk}(G)$ le rang maximal d'une matrice ayant même support que A. Il a été conjecturé [C. van Nuffelen, Amer. Math. Monthly 83 (1976) 265–266] que pour tout graphe G, $\chi(G) \leqslant \mathrm{rk}(G)$. Le premier contre-exemple à cette conjecture a été obtenu par Alon et Seymour [J. Graph Theor. 13 (1989) 523–525]. Récemment, Fishkind et Kotlov [Discrete Math. 250 (2002) 253–257] ont montré que pour tout graphe G, $\chi(G) \leqslant \mathrm{Rk}(G)$. Dans cette Note, nous améliorons cette borne et montrons $\chi(G) \leqslant \frac{\mathrm{rk}(G)+\mathrm{Rk}(G)}{2}$. ***Pour citer cet article : S. Akbari, H.-R. Fanaï, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Determining a good upper bound for the (vertex) chromatic number of a graph is a very important problem in graph theory and has been studied extensively over the years. In 1976 van Nuffelen conjectured [4] that the

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chromatic number of any graph with at least one edge does not exceed the rank of its adjacency matrix. The first counterexample to this conjecture was obtained by Alon and Seymour [1]. They constructed a graph with chromatic number 32 and with an adjacency matrix of rank 29. It was proved by Kotlov and Lovász, [2], that the number of vertices in a twin free graph (a graph with no two vertices with the same set of neighbors) is $O((\sqrt{2})^r)$, where r is the rank of adjacency matrix. In [3] it is shown that for any graph G with a nonempty edge set the chromatic number of G is, at most, the term rank of G, and one has equality if and only if (besides isolated vertices) G is the complete graph K_n or the star $K_{1,n-1}$. In the present Note we improve this upper bound and show that the chromatic number of a graph does not exceed the average of rank and term rank. Equality holds if and only if (besides isolated vertices) G is the complete graph K_n or the star $K_{1,n-1}$.

Before proving our results let us introduce some necessary notation. A subset X of the vertices of G is called a *clique* if the induced subgraph on X is a complete graph. We denote the size of the maximum clique of G by $\omega(G)$. A subset S of $V(G)$ is called an *independent set* of G if no vertices of S are adjacent in G. For a graph G, the maximal number $\alpha(G)$ of independent vertices is called the *independence number* of G.

A *k-vertex colouring* of a graph G is an assignment of k colours $\{1,\ldots,k\}$ to the vertices of G such that no two adjacent vertices have the same colour. The *vertex chromatic number* $\chi(G)$ of a graph G is the smallest k for which G has a k-vertex colouring. For any graph G of order n, the *adjacency matrix* of G is the $n\times n$ matrix $A(G)$ whose (i,j)th entry is 1 if $v_i \neq v_j$ are adjacent and 0 otherwise. The *rank* of G, denoted $\mathrm{rk}(G)$, is the rank of the adjacency matrix of G over $\mathbb{R}$. The *term rank* of G, denoted $\mathrm{Rk}(G)$, is the maximal rank of real $n\times n$ matrix with support (non-zero entries) included in the support of $A(G)$. A *two-factor* in G is a collection of vertex-disjoint cycles covering every vertex of G; here a single edge is considered a two-cycle. For a proof of the following result see, for instance, [5].

Lemma 1.1. *For any graph G with a nonempty edge set,* $\mathrm{Rk}(G)$ *is the maximum number of vertices in a subgraph H of G such that H has a two-factor.*

2. Results

The following theorem is a generalization of Corollary 1 of [3].

Theorem 2.1. *For any graph G with a nonempty edge set, $\chi(G) \leqslant \frac{\mathrm{Rk}(G)+\omega(G)}{2}$. Moreover, equality holds if and only if (besides isolated vertices) G is the complete graph K_n or the star $K_{1,n-1}$.*

Proof. Clearly we may assume that G is a graph with no isolated vertex. Let H be a maximal subgraph of G which has a two-factor. If H is a complete graph, then $\chi(G)=\chi(H)=\mathrm{Rk}(G)$. Clearly $\omega(G)=|V(H)|=\mathrm{Rk}(G)$ and the inequality holds. Thus suppose that H is not a complete graph. Assume that X_1 is an independent subset of $V(H)$ realizing the independence number $\alpha(H)=|X_1|$ of H. If $H\setminus X_1$ is a complete graph, then we set $\omega_1=|V(H\setminus X_1)|$, otherwise assume that X_2 is an independent subset of $V(H\setminus X_1)$ with maximal size. If we continue this procedure, we obtain $V(H)=(\bigcup_{i=1}^{k} X_i)\cup V(L)$, where $|X_i|\geqslant 2$ and L is a complete graph of size ω_1 (we note that L may be the empty graph). Now we colour each vertex in X_i with colour i, for i, $1\leqslant i\leqslant k$, and we colour all vertices of L with ω_1 new colours. We have

$$\chi(H)\leqslant \frac{\mathrm{Rk}(G)-\omega_1}{2}+\omega_1\leqslant \frac{\mathrm{Rk}(G)+\omega(G)}{2}.$$

Now if $\chi(G)=\chi(H)$, we are done. Thus assume that $\chi(G)>\chi(H)$. Since $V(G\backslash H)$ is independent, there exists a vertex $x\in V(G\setminus H)$ such that its degree in G is at least $\chi(H)$. Consider a two-factor of H as cycles or single edges. Since H is a maximal subgraph of G which has a two-factor, it is easy to verify that x cannot be adjacent to a vertex of an odd cycle of the two-factor of H. On the other hand, every even cycle has a two-factor which is a matching, so x is adjacent to at most half of the vertices of an even cycle of the two-factor of H, because H is a maximal subgraph of G which has a two-factor. This means that the degree of x in G is at most $\frac{\mathrm{Rk}(G)}{2}$. Combining these inequalities we obtain $\chi(H) \leqslant \frac{\mathrm{Rk}(G)}{2}$. Hence $\chi(G) \leqslant \frac{\mathrm{Rk}(G)}{2} + 1 \leqslant \frac{\mathrm{Rk}(G)+\omega(G)}{2}$ and the inequality is proved.

Now we show that if equality holds, then G is the complete graph K_n or the star $K_{1,n-1}$. We consider the following two cases:

Case 1. $\chi(G) = \chi(H)$. As we saw before, we have

$$\frac{\mathrm{Rk}(G)+\omega(G)}{2} = \chi(G) = \chi(H) \leqslant \frac{\mathrm{Rk}(G)-\omega_1}{2} + \omega_1 \leqslant \frac{\mathrm{Rk}(G)+\omega(G)}{2}.$$

This implies that $\omega_1 = \omega(G) \geqslant 2$ and $|X_i| = 2$, for any i. So $\alpha(H) \leqslant 2$. If H is a complete graph, then either $G = H = L$ is a complete graph or $G \neq H$. In the later case, since H is a maximal subgraph of G which has a two-factor and H is complete, we have $H = K_2$ and G must be a star. Thus suppose that H is not a complete graph. Hence we have $\mathrm{Rk}(G) = 2k + \omega(G)$, where $k \geqslant 1$ denotes the number of X_i's defined in the proof of the main inequality. Hence $\chi(G) = k + \omega(G)$. Now if we colour the vertices of X_i, by the colour i, for i, $1 \leqslant i \leqslant k-1$, and we colour all vertices of the complete graph L by $\omega(G)$ new colours, then we can colour two vertices of X_k with colours already used in the vertex colouring of L. This shows that $\chi(H) \leqslant k - 1 + \omega(G)$ which contradicts, $\chi(H) = \chi(G) = k + \omega(G)$.

Case 2. $\chi(G) \neq \chi(H)$. As we saw before, in this case we have

$$\frac{\mathrm{Rk}(G)+\omega(G)}{2} = \chi(G) \leqslant \frac{\mathrm{Rk}(G)}{2} + 1.$$

Hence $\omega(G) = 2$, $\chi(H) = \frac{\mathrm{Rk}(G)}{2}$ and $|V(H)| = \mathrm{Rk}(G)$ is an even number.

If $|X_1| \geqslant 4$, using the fact that $\omega_1 \leqslant 2$, we can colour $V(X_1 \cup L)$, with at most two colours. Therefore by the same argument used in the proof of the main inequality, we have $\chi(H) \leqslant \frac{\mathrm{Rk}(G)-5}{2} + 2$, a contradiction. It follows that $\alpha(H) \leqslant 3$. On the other hand, since $\chi(G) > \chi(H) = \frac{\mathrm{Rk}(G)}{2}$, there is a vertex $x \in V(G \setminus H)$ such that x is adjacent to at least $\frac{\mathrm{Rk}(G)}{2}$ vertices of H. Since $\omega(G) = 2$, these $\frac{\mathrm{Rk}(G)}{2}$ vertices of H are independent and noting that $\alpha(H) \leqslant 3$, we conclude that $|V(H)| = \mathrm{Rk}(G) \leqslant 6$. Now if H is not a bipartite graph, $|V(H)| = 6$ and H has a cycle C with length 5. If $y \in V(H \setminus C)$, by maximality of H, one can see that x is not adjacent to y. Thus x is adjacent to at least three vertices of C. This implies that $\omega(G) > 2$, a contradiction. Thus H is a bipartite graph and $2 = \chi(H) = \frac{\mathrm{Rk}(G)}{2}$. Hence G has no odd cycle and so it is a bipartite graph. Thus $\chi(G) = 2$ which contradicts $\chi(G) > \chi(H) = 2$. □

We note that the following result of Fishkind and Kotlov [3, Corollary 1] is an immediate consequence of Theorem 2.1.

Corollary 2.2. *For any graph G with a nonempty edge set, $\chi(G) \leqslant \mathrm{Rk}(G)$. Moreover, equality holds if and only if (besides isolated vertices) G is the complete graph K_n or the star $K_{1,n-1}$.*

Proof. Clearly for any graph G, $\omega(G) \leqslant \mathrm{Rk}(G)$, so $\chi(G) \leqslant \frac{\mathrm{Rk}(G)+\omega(G)}{2} \leqslant \mathrm{Rk}(G)$. On the other hand if $\chi(G) = \mathrm{Rk}(G)$, then we have $\mathrm{Rk}(G) = \omega(G)$ and this means that G is a complete graph or a star. □

Corollary 2.3. *For any graph G with a nonempty edge set, $\chi(G) \leqslant \frac{\mathrm{rk}(G)+\mathrm{Rk}(G)}{2}$. Moreover, equality holds if and only if (besides isolated vertices) G is the complete graph K_n or the star $K_{1,n-1}$.*

Proof. First note that the adjacency matrix of the complete graph K_r is of full rank. We have $A(K_r) = J - I$, where J is the $r \times r$ matrix with all entries 1. Thus $A(K_r)$ has $r-1$ eigenvalues -1 and a simple eigenvalue $r-1$, and is invertible for $r > 1$. It follows that $\chi(G) \leqslant \frac{\omega(G)+\mathrm{Rk}(G)}{2} \leqslant \frac{\mathrm{rk}(G)+\mathrm{Rk}(G)}{2}$. □

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References

[1] N. Alon, P. Seymour, A Counterexample to the rank-coloring conjecture, J. Graph Theor. 13 (1989) 523–525.
[2] A. Kotlov, L. Lovász, The rank and size of graphs, J. Graph Theor. 23 (1996) 185–189.
[3] D.E. Fishkind, A. Kotlov, Rank, term rank, and chromatic number, Discrete Math. 250 (2002) 253–257.
[4] C. van Nuffelen, A bound for the chromatic number of a graph, Amer. Math. Monthly 83 (1976) 265–266.
[5] E.R. Scheinerman, D.H. Ulman, Fractional Graph Theory, A Rational Approach to the Theory of Graphs, Wiley, New York, 1997.