Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 340 (2005) 225–228

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Probability Theory

On the first crossing times of a Brownian motion and a family of continuous curves

Larbi Alili [a], Pierre Patie [b]

[a] *Department of Statistics, The University of Warwick, Coventry CV4 7AL, UK*
[b] *RiskLab, Department of Mathematics, ETH-Zentrum, CH-8092 Zürich, Switzerland*

Received 21 September 2004; accepted after revision 3 November 2004

Available online 5 January 2005

Presented by Marc Yor

Abstract

We review the analytic transformations allowing to construct standard Brownian bridges from a Brownian motion. These are generalized and some of their properties are studied. The new family maps the space of continuous positive functions into a family of curves which is the topic of our study. We establish a simple and explicit formula relating the distributions of the first hitting times of each of these curves by a standard Brownian motion. ***To cite this article: L. Alili, P. Patie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur les premiers instants de croisement du mouvement brownien et d'une famille de courbes continues. Nous examinons les transformations analytiques qui permettent de passer du mouvement brownien aux ponts browniens standards. Nous les généralisons et étudions certaines de leurs propriétés. L'image d'une courbe réelle et continue, par ces transformations, est une famille de courbes à laquelle nous nous intéressons. Nous établissons une relation simple et explicite entre les distributions des temps d'atteinte de chacun des éléments de cette famille par un mouvement brownien. ***Pour citer cet article : L. Alili, P. Patie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Let B be a standard Brownian motion and f a continuous function on $\mathbb{R}^+$ such that $f(0) \neq 0$. The determination of the distribution of $T^{(f)} = \inf\{s \geqslant 0;\ B_s = f(s)\}$, known as the boundary crossing problem, is the topic of our

E-mail addresses: l.alili@warwick.ac.uk (L. Alili), patie@math.ethz.ch (P. Patie).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.11.008

study herein. This problem originally attracted researchers because of its connections to sequential analysis, non-parametric tests and iterated logarithm law, see [6] and the references therein. As a general result, we recall that Strassen, in [11], showed that if f is continuously differentiable then the latter is absolutely continuous with respect to the Lebesgue measure with a continuous density. From the explicit viewpoint, some elaborated fine methods proved efficient each for specific elementary examples. For instance, the Bachelier–Lévy formula for the straight lines, Doob's transform for the square root boundaries [1], and the direct application of Girsanov's theorem for the quadratic functions [3,10]. In the general setting, the celebrated method of images allows us to solve the problem for a class of curves which are solutions, in the unknown x for a fixed t, of implicit equations of the type

$$h(x,t) \overset{(\mathrm{def})}{=} \int_0^\infty \mathrm{e}^{ux-\frac{u^2}{2}t}\vartheta(\mathrm{d}u) = a$$

where a is some fixed positive constant and ϑ is a positive σ-finite measure. The collected class of curves which can be treated using this tool must satisfy some criterions such as the concavity, see [5]. Another method which is worth to be mentioned, discovered by Durbin, see in [2], transforms the problem into the calculation of a conditional expectation. Further reading about asymptotic studies, essays and other recent applications can be found in [7,8] and the references therein. We shall show a new classification of curves consisting in associating to each curve a family of ones. More precisely, to a fixed real β we introduce the functional transformation $S^{(\beta)}$ defined via: $S^{(\beta)}: C(\mathbb{R}^+,\mathbb{R}) \to C([0,\zeta^{(\beta)}),\mathbb{R})$ with $S^{(\beta)}(f) = (1+\beta\cdot)f(\frac{\cdot}{1+\beta\cdot})$ where $\zeta^{(\beta)} = -\beta^{-1}$ if $\beta < 0$ and equals $+\infty$ otherwise. Note that when $\beta < 0$ the image by $S^{(\beta)}$ of a standard Brownian motion is a Brownian bridge of length $-\beta^{-1}$. Now, for a fixed $f \in C(\mathbb{R}^+,\mathbb{R})$ we collect the family of curves $\{S^{(\beta)}(f)(t),\ \beta\in\mathbb{R},\ t<\zeta^{(\beta)}\}$ and compare the distributions of the hitting times of each element by a standard Brownian motion.

2. On the family $\{S^{(\beta)}, \beta \in \mathbb{R}\}$ and Brownian bridges

First, we show how the studied family of mappings shows up in the process of the construction of the family of Brownian bridges from a given original Brownian motion B. Let $B^{(\mathrm{br})}$ be a standard Brownian bridge of length $T > 0$ and recall that the latter might be defined through its unique decomposition as a semimartingale in its own filtration that is $B_t^{(\mathrm{br})} = B_t - \int_0^t \frac{B_s^{(\mathrm{br})}}{T-s}\,\mathrm{d}s$, $t < T$. This linear equation, when integrated, provides the well-known realization $B_t^{(\mathrm{br})} = (T-t)\int_0^t \frac{\mathrm{d}B_s}{T-s}$, $t < T$. Another decomposition, with respect to the filtration $\sigma\{B_s,\ s<T\}\vee\sigma\{B_T\}$, is given by $B_t^{(\mathrm{br})} = B_t - tB_T/T$, for $t \leqslant T$. Next, the law of $B^{(\mathrm{br})}$ can be constructed as a Doob h-transform of that of B i.e. the Wiener measure. That is with $\beta = -T^{-1}$, here we have $\beta < 0$, $\zeta^{(\beta)} = -\beta^{-1} = T$ and denoting by $\mathbb{P}^{(\mathrm{br})}$ and $\mathbb{P}$, respectively, the laws of $B^{(\mathrm{br})}$ and B, we have, for any fixed $t < \zeta^{(\beta)}$, the absolute continuity relationship

$$\mathrm{d}\mathbb{P}^{(\mathrm{br})}_{|\mathcal{F}_t} = (1+\beta t)^{-1/2}\,\mathrm{e}^{\frac{\beta}{2}\frac{B_t^2}{1+\beta t}}\,\mathrm{d}\mathbb{P}_{|\mathcal{F}_t}.$$

The above observations extend readily to the case when β is any real and the analogue of the Brownian bridges is the family of Gauss–Markov processes of Ornstein–Uhlenbeck type denoted by $(U^{(\beta)},\ \beta\in\mathbb{R})$ defined for $\beta\in\mathbb{R}$ and any fixed $0\leqslant t<\zeta^{(\beta)}$ by $U^{(\beta)} = S^{(\beta)}(B^{(\beta)})$ where $B^{(\beta)}$ is the continuous martingale given by $B_t^{(\beta)} = \int_0^{t/(1-\beta t)} \frac{\mathrm{d}B_s}{1+\beta s}$, $t < \zeta^{(-\beta)}$. Next, we observe that

$$\left\langle \int_0^\cdot \frac{\mathrm{d}B_s}{1+\beta s}\right\rangle_t = \frac{t}{1+\beta t} \underset{t\to\zeta\beta}{\longrightarrow} \begin{cases} \dfrac{1}{\beta}, & \beta>0,\\ \infty, & \text{otherwise.}\end{cases}$$

Thus, if $\beta \leqslant 0$ then $(B^{(\beta)},\ t\geqslant 0)$ is a standard Brownian motion. Otherwise, we can extend the definition of $(B^{(\beta)},\ t\leqslant\zeta^{(-\beta)})$ as follows

$$B_t^{(\beta)} = \begin{cases} \displaystyle\int_0^{t/(1-\beta t)} \frac{\mathrm{d}B_s}{1+\beta s}, & t \leqslant \beta^{-1}, \\ \displaystyle\int_0^{1/\beta} \frac{\mathrm{d}B_s}{1+\beta s} + \widetilde{B}_{t-1/\beta}, & t > \beta^{-1}, \end{cases}$$

where $\widetilde{B}$ is another Brownian motion that is independent of B, so that $B^{(\beta)}$ becomes a standard Brownian motion on $\mathbb{R}^+$. For additional information on these topics we refer to [9]. We proceed by providing some elementary algebraic and analytic properties of the studied family of mappings.

Proposition 2.1. *We have the following items.*

(i) *For any* $\alpha, \beta \in \mathbb{R}$, *we have* $S^{(\alpha)} \circ S^{(\beta)} = S^{(\alpha+\beta)}$.
(ii) $(S^{(\beta)})_{\beta \geqslant 0}$ *is a semi-group.*
(iii) *For* $\mu \in \mathbb{R}$ *we have* $S^{(\beta)}(f + \mu\cdot) = S^{(\beta)}(f) + \mu\cdot$.
(iv) *For any fixed* $\beta \in \mathbb{R}$, $S^{(\beta)}$ *is a linear mapping which invariant subspace is given by the space of linear functions.*
(v) $S^{(\beta)}$ *preserves the concave and convex properties.*
(vi) *Let* $f \in C(\mathbb{R}^+, \mathbb{R})$ *be a concave and non-increasing function such that* $f(0) > 0$ *and let* t_0 *be the smallest root of the equation* $f(t) = 0$. *Then* $S^{(1/t_0)}(f)$ *is increasing on* $[0, t_0)$.

3. Switching from a curve to a family of curves

Fix $\beta \in \mathbb{R}$ and let $U^{(-\beta)}$ be defined as above. Introduce $H_f^{(-\beta)} = \inf\{s \geqslant 0;\ U_s^{(-\beta)} = f(s)\}$ and to simplify notations set $f^{(\beta)} = S^{(\beta)}(f)$. For $\beta > 0$, observe that the support of $H_f^{(-\beta)}$ is $[0, \beta^{-1}] \cup \{+\infty\}$. Similarly, for $\beta < 0$ we close the curve $f^{(\beta)}$ at the lifetime $-\beta^{-1}$. The observations of the last section allow us to relate the times $H_f^{(-\beta)}$ and $T^{(f^{(\beta)})}$ as follows.

Lemma 3.1. *We have* $\{H_f^{(-\beta)} = \zeta^{(-\beta)}\} = \{T^{(f^{(\beta)})} = \zeta^{(\beta)}\}$ *and hold true the identities*

$$H_f^{(-\beta)} \overset{(d)}{=} T^{(f^{(\beta)})} \big/ \big(1 + \beta T^{(f^{(\beta)})}\big) \quad \textit{and} \quad T^{(f^{(\beta)})} \overset{(d)}{=} H_f^{(-\beta)} \big/ \big(1 - \beta H_f^{(-\beta)}\big).$$

Next, observe that we can define the family $\{S^{(\beta)},\ \beta \in \mathbb{R}\}$ on the space of probability measures. For instance in the absolute continuous case, to the probability measure $\mu(\mathrm{d}t) = h(t)\,\mathrm{d}t$ we associate $S^{(\beta)}(\mu)(\mathrm{d}t) = S^{(\beta)}(h(t))\,\mathrm{d}t$. Now, we state the main result of this note.

Theorem 3.2. *We have, for any fixed* $t < \zeta^{(\beta)}$, *the relation*

$$\mathbb{P}(T^{(f^{(\beta)})} \in \mathrm{d}t) = \frac{1}{(1+\beta t)^{5/2}}\, \mathrm{e}^{-\frac{1}{2}\frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} S^{(\beta)}\big(\mathbb{P}(T^{(f)} \in \mathrm{d}t)\big). \tag{1}$$

We gathered below the families corresponding to the most studied elementary curves. For any couple of reals a and b, we have the correspondences.

f	$f^{(\beta)}$
$a+bt$	$a+(b+a\beta)t$
$\sqrt{1+2bt}$	$\sqrt{(1+\beta t)(1+(\beta+2b)t)}$
$(b+t)^2$	$((b+(1+\beta)t)^2)/(1+\beta t)$

Let us recover the well-known Bachelier–Lévy formula corresponding to the straight line $(a+\mu t,\ t\geqslant 0)$. By setting $f=a$ and $\beta=\mu/a$ we get that $f^{(\beta)}(t)=a+\mu t$. With $\mathbb{P}(T^{(a)}\in \mathrm{d}t)=\frac{|a|}{\sqrt{2\pi t^3}}\,\mathrm{e}^{-\frac{a^2}{2t}}\,\mathrm{d}t$ an immediate application of Theorem 3.2 provides then that $\mathbb{P}(T^{(a+\mu\cdot)}\in \mathrm{d}t)=\frac{|a|}{\sqrt{2\pi t^3}}\,\mathrm{e}^{-\mu a-\frac{\mu^2}{2}t-\frac{a^2}{2t}}\,\mathrm{d}t$. Next, we look at the distribution of $T_a^{(\lambda_1,\lambda_2)}=\inf\{s>0;\ B_s=a\sqrt{(1+\lambda_1 s)(1+\lambda_2 s)}\}$, where a and $\lambda_1<\lambda_2$ are some fixed reals. At a first stage, consider the case $\lambda_2=0$ and set $\lambda_1=\lambda$ and $T_a^{(\lambda,0)}=T_a^{(\lambda)}$. With $X_t=\mathrm{e}^{-\lambda t/2}\int_0^t \mathrm{e}^{\lambda s/2}\,\mathrm{d}B_s$ for $t\geqslant 0$ and $\eta_a=\inf\{s\geqslant 0;\ X_s=a\}$, we have $T_a^{(\lambda)}\overset{(d)}{=}\lambda^{-1}(\mathrm{e}^{\lambda\eta_a}-1)$, see [1], which is to combine with the distribution of η_a found in [4]. Next, observe that it is enough to consider only the case where a is positive since the other case can be recovered thanks to the symmetry of B. We conclude by using the easily proved formula $\mathbb{P}(T_a^{(\lambda)}\in \mathrm{d}t)=\frac{1}{1+\lambda t}\mathbb{P}(\eta_a\in \mathrm{d}\cdot)|_{\cdot=\frac{1}{\lambda}\log(1+\lambda t)}\,\mathrm{d}t,\ t\leqslant \zeta^{(\lambda)}$. If $\lambda_1<\lambda_2$ then, on $[0,\zeta^{(\lambda_1)}]$ which is the support of $T_a^{(\lambda_1,\lambda_2)}$ if λ_1 is positive and its finite support otherwise, we have $S^{(\lambda_1)}(\sqrt{1+(\lambda_2-\lambda_1)\cdot})=\sqrt{(1+\lambda_2\cdot)(1+\lambda_1\cdot)}$ and conclude by making use of Theorem 3.2.

References

[1] L. Breiman, First exit times from a square root boundary, in: Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, CA, 1965/66), vol. II: Contributions to Probability Theory, Part 2, 1967, pp. 9–16.

[2] J. Durbin, The first-passage density of a continuous Gaussian process to a general boundary, J. Appl. Probab. 22 (1985) 99–122.

[3] P. Groeneboom, Brownian motion with a parabolic drift and Airy functions, Probab. Theory Related Fields 81 (1) (1989) 79–109.

[4] J. Keilson, H.F. Ross, Passage time distributions for Gaussian Markov (Ornstein–Uhlenbeck) statistical processes, in: Selected Tables in Mathematical Studies, III, Amer. Math. Soc., Providence, RI, 1975, pp. 233–327.

[5] H.R. Lerche, Boundary Crossing of Brownian Motion. Its Relation to the Law of the Iterated Logarithm and to Sequential Analysis, Lecture Notes in Statist., vol. 40, Springer-Verlag, Berlin, 1986.

[6] A.A. Novikov, On estimates and the asymptotic bahaviour of non-exit probabilities of a Wiener process to a moving boundary, 1981, pp. 495–505.

[7] G. Peskir, Limit at zero of the Brownian first-passage density, Probab. Theory Rel. Fields 124 (1) (2002) 100–111.

[8] K. Pötzelberger, L. Wang, Boundary crossing probability for Brownian motion, J. Appl. Probab. 38 (2001) 152–164.

[9] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, vol. 293, third ed., Springer-Verlag, Berlin, 1999.

[10] P. Salminen, On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary, Adv. Appl. Probab. 20 (2) (1988) 411–426.

[11] V. Strassen, Almost sure behavior of sums of independent random variables and martingales, in: Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, CA, 1965/66), vol. II, 1967.