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Probability Theory/Partial Differential Equations

Ergodic properties of highly degenerate 2D stochastic Navier–Stokes equations

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Abstract

This Note presents the results from “Ergodicity of the degenerate stochastic 2D Navier–Stokes equation” by M. Hairer and J.C. Mattingly. We study the Navier–Stokes equation on the two-dimensional torus when forced by a finite dimensional Gaussian white noise and give conditions under which the system is ergodic. In particular, our results hold for specific choices of four-dimensional Gaussian white noise. ***To cite this article: M. Hairer, J.C. Mattingly, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Propriétés ergodiques des équations de Navier–Stokes 2D stochastiques, hautement dégénérées. Cette Note présente les résultats de l’article « Ergodicity of the degenerate stochastic 2D Navier–Stokes equation » par M. Hairer et J.C. Mattingly. Nous étudions l’équation de Navier–Stokes sur le tore bidimensionel, excitée par un bruit blanc gaussien de dimension finie. Nous donnons des conditions suffisantes pour que la solution soit ergodique. Nos résultats sont en particulier vrais dans certains cas de bruit blanc gaussien de dimension quatre. ***Pour citer cet article : M. Hairer, J.C. Mattingly, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

This Note reports on recent progress made in [8] on the study of the two-dimensional Navier–Stokes equation driven by an additive stochastic forcing. These results make use in a critical fashion another set of recent results

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from Mattingly and Pardoux [16]. Recall that the Navier–Stokes equation describes the time evolution of an incompressible fluid. We quickly recall the formulation of the Navier–Stokes equation, referring the reader to [9] and the two source articles for more information. In vorticity form, the Navier–Stokes equation is given by

$$\frac{\partial w}{\partial t}(t,x) + B(w,w)(t,x) = \nu \Delta w(t,x) + \frac{\partial W}{\partial t}(t,x) \quad \text{and} \quad w(0,x) = w_0(x), \tag{1}$$

where $x = (x_1, x_2) \in \mathbb{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and $B(w, \widetilde{w})(x) = \sum_{i=1}^{2} (\mathcal{K}w)_i(x) \frac{\partial \widetilde{w}}{\partial x_i}(x)$, where $\mathcal{K}$ is the Biot–Savart integral operator which reconstructs the velocity from the vorticity. Its definition is given in [9]. As in [9], we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}^2_+ = \{(j_1, j_2) \in \mathbb{Z}^2\colon\ j_2 > 0\} \cup \{(j_1, j_2) \in \mathbb{Z}^2\colon\ j_1 > 0, j_2 = 0\}$, $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$ and $\mathbb{Z}^2_0 = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, we define a real Fourier basis for functions on $\mathbb{T}^2$ with zero spatial mean by $e_k(x)$ equals $\sin(k \cdot x)$ if $k \in \mathbb{Z}^2_+$ and $\cos(k \cdot x)$ if $k \in \mathbb{Z}^2_-$. Write $w(t, x) = \sum_{k \in \mathbb{Z}^2_0} \alpha_k(t) e_k(x)$ for the expansion of the solution in this basis. We solve (1) on the space $\mathbb{L}^2 = \{f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k\colon\ \sum |a_k|^2 < \infty\}$. For $f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k$, we define the norms $\|f\|^2 = \sum |a_k|^2$ and $\|f\|_1^2 = \sum |k|^2 |a_k|^2$. This Note emphasizes forcing which directly excites only a few degrees of freedom as it is both of primary modeling interest and is technically the most difficult. Specifically we take forcing of the form $W(t, x) = \sum_{k \in \mathcal{Z}_*} \sigma_k W_k(t) e_k(x)$. Here $\mathcal{Z}_*$ is a finite subset of $\mathbb{Z}^2_0$, $\sigma_k > 0$, and $\{W_k\colon\ k \in \mathcal{Z}_*\}$ is a collection of mutually independent standard scalar Brownian Motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As described in [9], the spread of the randomness through the system is captured by the sets $\mathcal{Z}_0 = \mathcal{Z}_* \cap (-\mathcal{Z}_*)$, $\mathcal{Z}_n = \{\ell + j \in \mathbb{Z}^2_0\colon\ j \in \mathcal{Z}_0,\ \ell \in \mathcal{Z}_{n-1}$ with $\ell^\perp \cdot j \neq 0,\ |j| \neq |\ell|\}$, and lastly, $\mathcal{Z}_\infty = \bigcup_{n=1}^\infty \mathcal{Z}_n$. $\mathcal{Z}_\infty$ captures the directions to which the randomness has spread. And as discussed in [9], its structure is related to the formal commutators of the infinite dimensional diffusion on $\mathbb{L}^2$ formally associated to (1). The Note [9] contains the results from Mattingly and Pardoux [16] on Malliavin calculus applied to (1) giving control of the smoothing of the probability transition density in the directions contained in span$\{e_k\colon\ k \in \mathcal{Z}_\infty\}$ and the following characterization of when $\mathcal{Z}_\infty = \mathbb{Z}^2_0$ from Hairer and Mattingly [8].

Proposition 1.1. *One has $\mathcal{Z}_\infty = \mathbb{Z}^2_0$ if and only if integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}^2_0$ and there exist at least two elements in $\mathcal{Z}_0$ with unequal euclidean norm.*

See [9] for further discussion of this proposition and related issues.

This Note describes results from Hairer and Mattingly [8], which uses the tools from [16] to build a theory which, when applied to (1), proves that it has a unique invariant measure under extremely general and essentially sharp assumptions. In addition to the tools from [16], they introduce a new concept and tool which together provide an abstract framework in which the ergodicity of (1) is proven. The concept is a generalization of the strong Feller property for a Markov process which, for reasons that will be made clear below, is called the *asymptotic strong Feller* property. The main feature of this property is that a diffusion which is irreducible and asymptotically strong Feller can have at most one invariant measure. It thus yields a natural generalization of Doob's theorem. The tool is an approximate integration by parts formula, in the sense of Malliavin calculus, which is used to prove that the system enjoys the asymptotic strong Feller property. To the best of the authors knowledge, this paper is the first to prove ergodicity of a nonlinear stochastic partial differential equation (SPDE) under assumptions comparable to those assumed when studying finite dimensional stochastic differential equations. The ergodic theory of infinite dimensional stochastic systems, and SPDEs specifically, has been a topic of intense study over the last two decades. Until recently, the forcing was always assumed to be elliptic and spatially rough. In our context this translates to $\mathcal{Z}_* = \mathbb{Z}^2_0$ and $|\sigma_k| \sim |k|^{-\alpha}$ for some positive α. Flandoli and Maslowski [6] first proved ergodic results for (1) under such assumptions. This line of inquiry was extended and simplified in [4]. They represent a larger body of literature which characterizes the extent to which classical ideas developed for finite dimensional Markov processes apply to infinite dimensional processes. Principally they use tools from infinite dimensional stochastic analysis to prove that the processes are strong Feller in an appropriate topology and then deduce ergodicity. Next three groups of authors in [10,1,18], contemporaneously greatly expanded the cases known to be ergodic. They use a Foias–Prodi type reduction, first adapted to the stochastic setting in [12] and the pathwise contraction of the high spatial frequencies already used in [13] to prove ergodicity of (1) at sufficiently high viscosity. All of these results hinged on the observation that stochastically perturbing all of the unstable directions implies unique ergodicity. A general overview and simple examples can be found in [15]. These ideas have been continued and developed further in a number of papers. See [17,2,14,7,11,15]. Unfortunately, the best current estimates on the number of unstable directions in (1) grow as ν^{-1}. Hence the physically important limit of $\nu \to 0$ while a fixed, finite scale is forced were previously outside the scope of the theory. However there existed strong indications that ergodicity held in this case as [19] showed that the generator of the diffusion associated to finite dimensional Galerkin approximations of (1) was hypoelliptic in the sense of Hörmander when only a few directions were forced. This hypoellipticity is the crucial ingredient in the proof of ergodicity from [19]. The 'correct' ergodic theorem needs to incorporate how the randomness spreads from the few forced directions to all of the unstable directions. Combining this understanding with was learned in [12,13,10,1,18] should yield unique ergodicity. This program is executed in the papers described in this Note.

2. Unique ergodicity

Recall that an *invariant measure* for (1) is a probability measure $\mu_\star$ on $\mathbb{L}^2$ such that $P_t^*\mu_\star = \mu_\star$, where P_t^* is the semigroup on measures dual to the Markov transition semigroup P_t defined by $(P_t\phi)(w) = \mathbb{E}_w\phi(w_t)$ with $\phi \in C_b(\mathbb{L}^2)$. While the existence of an invariant measure for (1) can be proved by 'soft' techniques using the regularizing and dissipativity properties of the flow [3,5], showing its uniqueness is a more challenging problem. If $\mu_\star$ is unique then $\frac{1}{T}\mathbb{E}\int_0^T \phi(w_t)\,\mathrm{d}t \to \int_{\mathbb{L}^2} \phi(w)\,\mu_\star(\mathrm{d}w)$ as $T \to \infty$, for all bounded continuous functions ϕ and all initial conditions $w_0 \in \mathbb{L}^2$. It thus gives mathematical support to the *ergodic assumption* usually made in the physics literature when discussing the qualitative behavior of (1). The following theorem is the main result of [8].

Theorem 2.1. *If $\mathcal{Z}_\infty = \mathbb{Z}_0^2$, then* (1) *has a unique invariant measure in* $\mathbb{L}^2$.

Combining, this theorem and Proposition 1.1 produces a simple criteria ensuring a unique invariant measure. The concept of a strong Feller Markov process appears to be less useful in infinite than finite dimensions. Specifically, if P_t is strong Feller and irreducible, then the measures $P_t(u,\cdot)$ and $P_t(v,\cdot)$ are equivalent for all initial conditions $u, v \in \mathbb{L}^2$. It is easy to construct an ergodic SPDE which does not satisfy this property. Recall the following sufficient criteria for P_t to be strong Feller: $|\nabla(P_t\phi)(w)| \leqslant C(w,t)\|\phi\|_\infty$ for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$ and a fixed locally bounded function C. While we will not give the exact definition of the asymptotic strong Feller property here, the following similar condition implies that the process is asymptotically strong Feller: there exists a locally bounded $C(w)$, a non-decreasing sequence of times t_n, and a strictly decreasing sequence ϵ_n with $\epsilon_n \to 0$ so that

$$\left|\nabla(P_{t_n}\phi)(w)\right| \leqslant C(w)\|\phi\|_\infty + \epsilon_n\|\nabla\phi\|_\infty \tag{2}$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$ and all $n \geqslant 1$. Since in applications typically $t_n \to \infty$, the process behaves as if it acquired the strong Feller property at time infinity, justifying the term asymptotic strong Feller. First, the chain rule implies that $\langle \nabla_w(P_t\phi)(w), \xi\rangle = \mathbb{E}_w(\nabla\phi)(w_t)J_{0,t}\xi$, where $J_{0,t}$ denotes the Jacobian for the solution flow at time t. Next we seek a direction v in the Cameron–Martin space so that if $\mathcal{D}^v$ denotes the Malliavin derivative in the direction v then $J_{0,t}\xi = \mathcal{D}^v w_t$. While often possible in finite dimensions, in infinite dimensions we know only how to achieve this up to some error. Setting $\rho_t = J_{0,t}\xi - \mathcal{D}^v w_t$, we have the approximate integration by parts formula:

$$\mathbb{E}_w(\nabla\phi)(w_t)J_{0,t}\xi = \mathbb{E}_w\mathcal{D}^v\big[\phi(w_t)\big] + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t = \mathbb{E}_w\phi(w_t)\int_0^t v_s\,\mathrm{d}W_s + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t.$$

This equality quickly implies (2), provided $\mathbb{E}|\int_0^\infty v_s\,\mathrm{d}W_s| < \infty$ and $\mathbb{E}|\rho_t| \to 0$ as $t \to \infty$. In [8], a v_t is chosen so that these conditions hold. The analysis is complicated by the fact that the v_t used there is not adapted to the Brownian filtration. This complication seems unavoidable. Hence, the stochastic integral is a Skorohod integral which complicates all of the calculations. Specifically, on intervals of the form $[n, n+\frac{1}{2}]$ with $n \geqslant 0$ in $\mathbb{N}$, we take v to be the least squares solution to minimizing $\|\rho_{n+1/2}\|^2$ where the norm of v is measured in metric induced by the regularized version of the inverse of the Malliavin matrix obtained by adding a small multiple of the identity. This has the effect of choosing v on $[n, n+\frac{1}{2}]$ to cancel the large scale components of $\rho_{n+1/2}$. Setting v equal to zero on intervals of the form $(n+\frac{1}{2}, n+1)$ controls the small scale components since the unforced linearized small scale dynamics are contractive. The mixing of this pathwise contractivity in the high modes and the probabilistic smoothing in the lower modes has its origins in the Gibbsian reductions and High/Low splitting of [12,10,1,18]. Theorem 3.2 from [9], whose proof is found in [16], is critical to ensure that cancelling the large scales of ρ with the variation v does not increase the small scales of ρ substantially. These techniques can also yield exponential mixing using the ideas from [14,7]. These results will be presented elsewhere.

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