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Numerical Analysis

Various characterisations of Extended Chebyshev spaces via blossoms

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Abstract

Among all W-spaces (i.e. spaces with nonvanishing Wronskians), extended Cheyshev spaces can be characterised by the existence of either Bernstein bases, or B-spline bases, or Bézier points, or blossoms in the spaces obtained by integration. ***To cite this article: M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Quelques caractérisations des espaces de Chebyshev généralisés liées à la notion de floraison. Parmi les W-espaces (espaces à Wronskiens sans zéro), les espaces de Chebyshev généralisés se caractérisent par l'existence de bases de Bernstein, ou de points de Bézier, ou de floraisons, ou de bases de B-splines, dans l'espace obtenu par intégration. ***Pour citer cet article : M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Points de Bézier et bases de Bernstein, pôles d'une spline et bases de B-splines, sont des notions fondamentales en Design Géométrique, qui toutes s'articulent de façon naturelle autour de la notion de *floraison*. Mettant à profit quelques résultats obtenus récemment, nous faisons ici le point sur les liens qui unissent ces notions, et comment elles sont reliées aux espaces de Chebyshev généralisés qu'elles permettent de caractériser de la manière suivante.

Théorème 0.1. *Soit $\mathcal{U}$ un espace de dimension n formé de fonctions C^{n-1} sur un intervalle I, et $\mathcal{E}$ le sous-espace de dimension $(n+1)$ de $C^n(I)$ défini par $\mathcal{E} := \{U \in C^n(I) \mid U' \in \mathcal{U}\}$. Les propriétés suivantes sont équivalentes :*

(i) *$\mathcal{U}$ est un espace de Chebyshev généralisé sur I ;*
(ii) *$\mathcal{E}$ est un W-espace sur I (au sens où le Wronskien d'une base ne s'annule en aucun point de I) et les floraisons existent dans $\mathcal{E}$;*

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(iii) *tout élément de l'espace $\mathcal{E}$ admet des points de Bézier par rapport à n'importe quel couple de points distincts de I ;*
(v) *l'espace $\mathcal{E}$ admet une base de Bernstein par rapport à n'importe quel couple de points distincts de I ;*
(vi) *tout espace de splines basé sur $\mathcal{E}$ possède une base de B-splines.*

Les définitions précises des différentes notions seront données dans les paragraphes suivants. Les preuves des équivalences précédentes reposent essentiellement sur deux propriétés : d'une part la caractérisation des espaces de Chebyshev généralisés en termes de bases *de type Bernstein*, i.e., de bases ayant des zéros prescrits en deux points distincts donnés à l'avance ; d'autre part la pseudoaffinité des floraisons en chaque variable. De cette dernière propriété découle aussi automatiquement la totale positivité et l'optimalité des bases en question.

1. The tools

We denote by I a real interval with a nonempty interior. We assume that $(U_0, \ldots, U_n)$ is a basis of a given $(n+1)$-dimensional space $\mathcal{F}$ of C^n functions $(n \geqslant 1)$, and we set $\mathbf{U}(x) := (U_0(x), \ldots, U_n(x))^{\mathrm{T}}$ for all $x \in I$.

1.1. Some interesting bases

Writing a function $F \in \mathcal{F}^d$ $(d \geqslant 1)$ as $F(x) = \sum_{i=0}^{n} U_i(x) P_i$, the points $P_0, \ldots, P_n \in \mathbb{R}^d$ are called *the control points of F relative to the basis* $(U_0, \ldots, U_n)$, and the polygon $[P_0, \ldots, P_n]$ its *control polygon*. In geometric design it is usual to use a certain type of bases as recalled below.

Definition 1.1. A basis $(U_0, \ldots, U_n)$ of $\mathcal{F}$ is said to be *normalised* if $\sum_{i=0}^{n} U_i = \mathbb{1}$. It is said to be *totally positive* (on I) if, for any sequence of points $x_0 < x_1 < \cdots < x_n$ in I, the matrix $(U_j(x_i))_{0 \leqslant i,j \leqslant n}$ is totally positive (i.e., all its minors are nonnegative).

Relative to a Normalised Totally Positive (NTP) basis, the control polygon of a curve provides interesting information on the curve. Firstly, the curve is contained in the convex hull of its control polygon. Secondly, no affine line can intersect the curve more times than it intersects its control polygon. It follows that the curve inherits properties from its control polygon: this is known as *shape preservation* properties of NTP bases. This is the reason why such bases are of so fundamental importance in geometric design.

Theorem 1.2 [1]. *If the space $\mathcal{F}$ possesses an NTP basis, then it possesses an optimal one, that is, the maximum element of the set of its NTP bases, relative to the ordering $(V_0, \ldots, V_n) \prec (U_0, \ldots, U_n)$ if there exists a stochastic totally positive matrix A such that $(V_0, \ldots, V_n) = (U_0, \ldots, U_n)A$.*

The optimal NTP basis is the basis which guarantees the best shape preserving properties. As an instance, the Bernstein basis

$$B_i^n(x) := \binom{n}{i}(1-x)^{n-i}x^i, \quad 0 \leqslant i \leqslant n,$$

is the optimal NTP basis of the polynomial space of degree n restricted to $[0, 1]$. Inspired by the latter example, it is natural to introduce more generally Bernstein bases as follows.

Definition 1.3. Given $a, b \in I$, $a \neq b$, we say that $(U_0, \ldots, U_n)$ is *the Bernstein basis relative to* (a, b) if the functions following properties are satisfied:

$(\mathrm{BB})_1$ for $0 \leqslant i \leqslant n$, U_i vanishes exactly i times at a and exactly $(n-i)$ times at b;
$(\mathrm{BB})_2$ $(U_0, \dots, U_n)$ is normalised;
$(\mathrm{BB})_3$ for $0 \leqslant i \leqslant n$, $\mathcal{B}_i$ is positive strictly between a and b.

If $(U_0, \dots, U_n)$ satisfies only $(\mathrm{BB})_1$ we say that it is a *Bernstein-like basis relative to* (a, b).

1.2. W-spaces, EC-spaces

Definition 1.4. The space $\mathcal{F}$ is said to be a *W-space on* I if the Wronskian $W(U_0, \dots, U_n)(x) := \det(\mathbf{U}(x), \mathbf{U}'(x), \dots, \mathbf{U}^{(n-1)}(x))$ never vanishes on I. It is said to be *an Extended Chebyshev space on* I (EC-space), if for any $r \geqslant 1$, any positive numbers $\mu_1, \dots, \mu_r$ with $\sum_{i=1}^r \mu_i = n$, any pairwise distinct $\tau_1, \dots, \tau_r \in I$, the determinant $\det(\mathbf{U}(\tau_1), \dots, \mathbf{U}^{(\mu_1-1)}(\tau_1), \dots, \mathbf{U}(\tau_r), \dots, \mathbf{U}^{(\mu_r-1)}(\tau_r))$ is different from zero.

In any space of C^n functions, multiple zeros can be counted up to order $(n+1)$. The space $\mathcal{F}$ is an EC-space on I iff any nonzero element of $\mathcal{F}$ vanishes at most n times on I, counting multiplicities. This makes EC-spaces the natural generalisations of polynomial spaces (see [2,10] for a thorough presentation of these spaces).

Obviously, if $\mathcal{F}$ is an EC-space on I, it possesses a Bernstein-like basis relative to any pair of distinct points of I. We proved the converse in [7], so that

Theorem 1.5. *The space $\mathcal{F}$ is an EC-space on I if and only if it possesses a Bernstein-like basis relative to any pair of distinct points of I.*

If the space $\mathcal{F}$ is an EC-space on I, supposed to contain constants, according to the latter theorem, it possesses Bernstein-like bases relative to any pairs of distinct points of I, but not necessarily normalised ones (e.g., the space spanned by $1, \cos x, \sin x$ over $[0, \pi]$). Hence it does not necessarily possesses Bernstein bases. It may also occur that a normalised Bernstein-like basis does exist, but without satisfying the positivity requirement (e.g., the same space over $[0, \alpha]$ with $\pi < \alpha < 2\pi$).

1.3. Spline spaces and B-spline bases

Subsequently, given a real number x and a positive integer μ, we use the notation $x^{[\mu]}$ with the meaning of x repeated μ times.

Given a sequence of points $t_0 < t_1 < \cdots < t_q < t_{q+1}$ in the interval I, to each t_i we allocate a multiplicity m_i, with $m_0 = m_{q+1} = n+1$ and $1 \leqslant m_i \leqslant n$ for $1 \leqslant i \leqslant q$. We then define the corresponding *knot vector* as the $(m+2n+2)$-tuple (with $m := \sum_{i=1}^q m_i$)

$$\mathcal{K} := (\xi_{-n}, \dots, \xi_{m+n+1}) := ({t_0}^{[m_0]}, {t_1}^{[m_1]}, \dots, {t_q}^{[m_q]}, {t_{q+1}}^{[m_{q+1}]}). \tag{1}$$

Associated with the latter knot vector, consider the *spline space based on* $\mathcal{F}$, defined as the set $\mathcal{S}(\mathcal{F}, \mathcal{K})$ of all functions $S : [t_0, t_{q+1}] \to \mathbb{R}$ such that

(a) for $1 \leqslant i \leqslant q$, S is C^{n-m_i} at t_i;
(b) for $0 \leqslant i \leqslant q$, there exists a function $F_i \in \mathcal{F}$ such that

$$S(x) = F_i(x), \quad x \in [t_i, t_{i+1}]. \tag{2}$$

Note that the space $\mathcal{F}$ is automatically contained in any spline space $\mathcal{S}(\mathcal{F}, \mathcal{K})$. On the other hand, due to $\mathcal{F}$ being $(n+1)$-dimensional and to the continuity conditions (a), the spline space $\mathcal{S}(\mathcal{F}, \mathcal{K})$ is of dimension at least $(q+1)(n+1) - \sum_{i=1}^q (n - m_i + 1) = m + n + 1$. It is exactly of dimension $(m+n+1)$ for instance as soon as $\mathcal{F}$ is a W-space on I.

Definition 1.6. When $\mathcal{S}(\mathcal{F},\mathcal{K})$ is $(m+n+1)$-dimensional, we say $\mathcal{N}_{-n},\dots,\mathcal{N}_m \in \mathcal{S}(\mathcal{F},\mathcal{K})$ form a *B-spline basis* if they satisfy the following properties:

(BSB)$_1$ support property: for all $j \in \{-n,\dots,m\}$, $\mathcal{N}_j$ has support $[\xi_j, \xi_{j+n+1}]$;
(BSB)$_2$ end points property: for all $j \in \{-n,\dots,m\}$, the function $\mathcal{N}_j$ vanishes exactly $(n-m_k+p+1)$ times at the left end point $\xi_j = t_k$ of its support, and exactly $(n-m_{k'}+p'+1)$ times at its right end point $\xi_{j+n+1} = t_{k'}$, where

$$p := \sharp\{\ell < j \mid \xi_\ell = \xi_j\}, \qquad p' := \sharp\{\ell > j+n+1 \mid \xi_\ell = \xi_{j+n+1}\};$$

(BSB)$_3$ normalisation property: $\sum_{j=-n}^m \mathcal{N}_j(x) = 1$ for all $x \in I$;
(BSB)$_4$ positivity property: for all $j \in \{-n,\dots,m\}$, $\mathcal{N}_j$ is positive on the interior of its support.

If the basis $(\mathcal{N}_{-n},\dots,\mathcal{N}_m)$ satisfies only (BSB)$_1$ and (BSB)$_2$, we say that it is a *B-spline-like basis* of $\mathcal{S}(\mathcal{F},\mathcal{K})$.

The previous properties ensure the linear independence of the $\mathcal{N}_j$'s, hence the fact they form a basis. Observe that, for $q=0$, the $(n+1)$ functions $(\mathcal{N}_{-n},\dots,\mathcal{N}_0)$ form a B-spline-like basis (resp. the B-spline basis) of the space $\mathcal{S}(\mathcal{F},\mathcal{K})$ iff the functions $\mathcal{B}_0,\dots,\mathcal{B}_n$, defined by $\mathcal{B}_i := \mathcal{N}_{i-n}$, form a Bernstein-like basis (resp. the Bernstein basis) relative to (t_0,t_1) in the space $\mathcal{F}$ restricted to $[t_0,t_1]$.

2. Bernstein bases, B-spline bases: existence, optimality

From now on, we denote by $\mathcal{U}$ a given n-dimensional space of C^{n-1} functions ($n \geqslant 1$) defined on I. Associated with $\mathcal{U}$ we consider the space $\mathcal{E}$ obtained by integration, namely

$$\mathcal{E} := \big\{U \in C^n(I) \mid U' \in \mathcal{U}\big\}. \tag{3}$$

Obviously, $\mathcal{E}$ is $(n+1)$-dimensional and it contains constants. Moreover, from the equality $W(\mathbb{1}, U_1,\dots,U_n) = W(U_1',\dots,U_n')$ and from Rolle's theorem, we have

$$\mathcal{U} \text{ is a W-space on } I \quad \Leftrightarrow \quad \mathcal{E} \text{ is a W-space on } I;$$
$$\mathcal{U} \text{ is an EC-space on } I \quad \Rightarrow \quad \mathcal{E} \text{ is an EC-space on } I.$$

We shall see in Subsection 2.2 how intimately Bernstein bases, B-spline bases, and EC-spaces are connected. *Blossoms* (see [9,8]) are the most efficient and elegant tools to clearly establish the links between them.

2.1. Blossoms

Select n functions $\Phi_1,\dots,\Phi_n \in \mathcal{E}$ so that $(\mathbb{1},\Phi_1,\dots,\Phi_n)$ form a basis of $\mathcal{E}$, or, equivalently, so that $(\Phi_1',\dots,\Phi_n')$ form a basis of $\mathcal{U}$. We consider the function $\Phi : I \to \mathbb{R}^n$ defined by

$$\Phi(x) := \big(\Phi_1(x),\dots,\Phi_n(x)\big)^{\mathrm{T}}, \quad x \in I. \tag{4}$$

The function Φ is the *mother-function* from which any function $F \in \mathcal{E}^d$ can be obtained via affine maps. All notions introduced below are independent of the selected mother-function. The osculating flat of any order $i \leqslant n$ at a point $x \in I$ is the affine flat going through $\Phi(x)$ and the direction of which is the linear space spanned by $\Phi'(x),\dots,\Phi^{(i)}(x)$. We denote it by $\mathrm{Osc}_i\,\Phi(x)$.

Definition 2.1. We say that blossoms exist in the space $\mathcal{E}$ if, for any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^r \mu_i = n$, and any $\tau_1 < \cdots < \tau_r$ in I, the osculating flats $\mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i)$, $1 \leqslant i \leqslant r$, intersect at a single point. The *blossom* of Φ is then the symmetric function $\varphi := (\varphi_1, \ldots, \varphi_n) : I^n \to \mathbb{R}^n$ such that:

$$\left\{\varphi({\tau_1}^{[\mu_1]}, \ldots, {\tau_r}^{[\mu_r]})\right\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i). \tag{5}$$

For any affine map $h : \mathbb{R}^n \to \mathbb{R}^d$, the blossom f of $F := h \circ \Phi$ is then defined as $f := h \circ \varphi$. Equalities involving blossoms can be derived by taking images under affine maps of the corresponding equalities on the blossom of the mother-function. This is the reason why we can limit ourselves to working with Φ and φ.

We say that an n-tuple $(x_1, \ldots, x_n) \in I^n$ is *admissible relative to* the knot vector $\mathcal{K}$ defined in (1) if, whenever $\min(x_1, \ldots, x_n) < t_\ell < \max(x_1, \ldots, x_n)$ for some integer ℓ, $1 \leqslant \ell \leqslant q$, at least m_l among the points $x_1, \ldots, x_n$ are equal to t_ℓ. We denote by $\mathcal{A}(\mathcal{K})$ the subset of I^n composed of all admissible n-tuples relative to $\mathcal{K}$. In the particular case $q = 0$, $\mathcal{A}(\mathcal{K}) = I^n$. Suppose that blossoms exist in the space $\mathcal{E}$. If $\mathcal{E}$ remains $(n+1)$-dimensional by restriction to any nontrivial subinterval of I (e.g., if $\mathcal{E}$ is a W-space on I), then the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is $(n+m+1)$-dimensional. Selecting a basis $(\mathbb{1}, \Sigma_1, \ldots, \Sigma_{n+m}) \in \mathcal{S}(\mathcal{E}, \mathcal{K})$, we consider the mother-spline function $\Sigma := (\Sigma_1, \ldots, \Sigma_{n+m})^{\mathrm{T}}$. Existence of blossoms in the space $\mathcal{E}$ implies existence of blossoms in the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$, not defined on the whole of I^n, but only on the set $\mathcal{A}(\mathcal{K})$ [5]. The blossom σ of Σ is the symmetric function such that, whenever the n-tuple $({\tau_1}^{[\mu_1]}, \ldots, {\tau_r}^{[\mu_r]})$ is admissible, with $\tau_1 < \cdots < \tau_r$ and positive μ_i's,

$$\left\{\sigma({\tau_1}^{[\mu_1]}, \ldots, {\tau_r}^{[\mu_r]})\right\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Sigma(\tau_i). \tag{6}$$

Due to the admissibility, all osculating flats $\mathrm{Osc}_{n-\mu_i}\, \Sigma(\tau_i)$ involved in (6) are well defined, except possibly for the first and last ones, which, if necessary, must be interpreted as $\mathrm{Osc}_{n-\mu_1}\, \Sigma(\tau_1^+)$ and $\mathrm{Osc}_{n-\mu_r}\, \Sigma(\tau_r^-)$, respectively.

2.2. The result

Theorem 2.2. *The two spaces $\mathcal{U}$ and $\mathcal{E}$ being linked by* (3)*, the following six properties are equivalent*:

(i) *the space $\mathcal{U}$ is an EC-space on I*;
(ii) *$\mathcal{E}$ is a W-space on I and blossoms exist in $\mathcal{E}$*;
(iii) *for any distinct $a, b \in I$, and any integer i, $0 \leqslant i \leqslant n$, the function Φ possesses $(n+1)$ Bézier points relative to (a, b), defined by*

$$\left\{\Pi_i(a, b)\right\} := \mathrm{Osc}_i\, \Phi(a) \cap \mathrm{Osc}_{n-i}\, \Phi(b), \quad 0 \leqslant i \leqslant n. \tag{7}$$

(iv) *$\mathcal{E}$ is a W-space on I and for any distinct $a, b \in I$ and any integer i, $1 \leqslant i \leqslant n-1$, the osculating flats* $\mathrm{Osc}_i\, \Phi(a)$ *and* $\mathrm{Osc}_{n-i}\, \Phi(b)$ *have a unique common point*;
(v) *the space $\mathcal{E}$ possesses a normalised Bernstein-like basis relative to any pair of distinct points of I*;
(vi) *for any knot vector $\mathcal{K}$, the space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ possesses a normalised B-spline-like basis.*

Moreover, when the latter properties are satisfied, then:

(vii) *relative to any given $(a, b) \in I^2$, $a \neq b$, the normalised Bernstein-like basis $(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ automatically satisfies the positivity property* $(\mathrm{BB})_3$*, i.e., it is the Bernstein basis. The control points of Φ relative to $(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ are its Bézier points, i.e.,*

$$\Phi(x) = \sum_{i=0}^{n} \mathcal{B}_i(x) \varphi(a^{[n-i]}, b^{[i]}), \quad x \in I. \tag{8}$$

If $a < b$, $(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ is the optimal NTP basis of the space $\mathcal{E}$ restricted to $[a, b]$.

(viii) *for any knot vector $\mathcal{K}$, the normalised B-spline-like basis automatically satisfies the positivity property* $(\mathrm{BSB})_4$, *i.e., it is the B-spline basis of the space $\mathcal{S}(\mathcal{E},\mathcal{K})$. It is also its optimal NTP basis. Relative to it, the control points of Σ are its poles, that is, the points $\sigma(\xi_{i+1},\dots,\xi_{i+n})$, $-n \leqslant i \leqslant m$, i.e.,*

$$\Sigma(x) = \sum_{i=-n}^{m} \mathcal{N}_i(x)\sigma(\xi_{i+1},\dots,\xi_{i+n}), \quad x \in [t_0, t_{q+1}]. \tag{9}$$

Sketch of the proof. First, note that (ii) $\Rightarrow$ (iv) is obvious, and that (v) can be considered a particular case of (vi) (see comments following Definition 1.6) and (vii) a particular case of (viii).

• (i) $\Rightarrow$ (ii). With no loss of generality, one can assume I to be closed and bounded. A classical result on EC-spaces says that $\mathcal{U}$ is the kernel of a differential operator of order n, associated with *weight functions*. Using these weight functions, existence of blossoms can then be derived as in [4].

• (iii) $\Leftrightarrow$ (iv) We just want to explain the slight difference between the two statements. The case $i = 0$ in (iii) means that, for any $b \in I$, the nth order osculating flat $\mathrm{Osc}_n\,\Phi(b)$ contains all points $\Phi(a)$, $a \in I$. Due to $(\mathbb{1}, \Phi_1,\dots,\Phi_n)$ being a basis of $\mathcal{E}$, this just means that, for any $b \in I$, $\mathrm{Osc}_n\,\Phi(b)$ is an n-dimensional affine flat, or, equivalently, that $\mathcal{U}$ is a W-space on I. The converse part is obvious.

• (i) $\Leftrightarrow$ (iv). Property (iv) is satisfied iff, for all distinct $a, b \in I$, and for $0 \leqslant i \leqslant n$, the n vectors $\Phi'(a),\dots,\Phi^{(i)}(a)$, $\Phi'(b),\dots,\Phi^{(n-i)}(b)$ are linearly independent. Clearly, this is also equivalent to existence of a Bernstein-like basis in the space $\mathcal{U}$, relative to any pair of distinct points of I. Hence, the equivalence between (i) and (iv) follows from Theorem 1.5.

• (v) $\Rightarrow$ (iv). From a normalised Bernstein-like basis relative to (a, b) in the space $\mathcal{E}$, it is easy to derive a Bernstein-like basis in the space $\mathcal{U}$ relative to the same (a, b) (see [7]).

• (i) $\Rightarrow$ (vi) and (viii). Suppose that (i) holds and consider the knot vector $\mathcal{K}$ given by (1). Not only do blossoms exist in the space $\mathcal{E}$, but they are *pseudoaffine* in each variable, in the sense that, for any $(x_1,\dots,x_{n-1}) \in I^{n-1}$, the point $\varphi(x_1,\dots,x_{n-1},x)$ moves in a strictly monotone way on an affine line as x moves in I (see [7] for a proof of this). Along with symmetry and with the fact that φ gives Φ by restriction to the diagonal of I^n, pseudoaffinity is fundamental to develop all geometric design algorithm. In the spline space $\mathcal{S}(\mathcal{E},\mathcal{K})$, blossoms also exist and they are pseudoaffine in each variable, but on the set $\mathcal{A}(\mathcal{K})$. This makes possible to develop an n-step *de Boor algorithm* to evaluate all values of Σ as convex combinations of the $(n+m+1)$ poles of Σ (defined in (viii)). This classically leads to (9), where the $\mathcal{N}_j$'s, $-n \leqslant j \leqslant m$, form a basis of $\mathcal{S}(\mathcal{E},\mathcal{K})$ and satisfy $(\mathrm{BSB})_i$, $i = 1, 3, 4$. It is an NTP basis because it emerges from the de Boor algorithm which is a so-called *corner cutting* algorithm. The fact that they also satisfy $(\mathrm{BSB})_2$ is due to the geometric meaning of the poles through (6) [3]. In addition, this guarantee that it is the optimal one [6]. □

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