Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 683–688
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

A note on the long time behavior for the drift-diffusion-Poisson system ☆

Naoufel Ben Abdallah, Florian Méhats, Nicolas Vauchelet

Mathématiques pour l'industrie et la physique, UMR 5640, université Paul Sabatier, UFR MIG, 118, route de Narbonne, 31032 Toulouse cedex 4, France

Received 6 May 2004; accepted after revision 27 September 2004

Available online 27 October 2004

Presented by Pierre-Louis Lions

Abstract

In this note we analyze the long time behavior of a drift-diffusion-Poisson system with a symmetric definite positive diffusion matrix, subject to Dirichlet boundary conditions. This system models the transport of electrons in semiconductor or plasma devices. By using a quadratic relative entropy obtained by keeping the lowest order term of the logarithmic relative entropy, we prove the exponential convergence to the equilibrium. ***To cite this article: N. Ben Abdallah et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une note sur le comportement en temps long du système dérive-diffusion-Poisson. Dans cette note, nous analysons le comportement en temps long des solutions du système couplé dérive-diffusion-Poisson avec une matrice de diffusion définie positive et soumis à des conditions aux limites de Dirichlet. Ce système modélise le transport de charges dans des dispositifs à semiconducteurs ou à plasmas. En utilisant l'entropie relative développée à l'ordre 2, nous prouvons la convergence exponentielle des solutions vers l'équilibre. ***Pour citer cet article : N. Ben Abdallah et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Dans cette note, nous analysons le comportement en temps long des solutions du système couplé dérive-diffusion-Poisson (2), (3), où la matrice de diffusion $\mathbb{D}$ est symétrique et définie positive, dans un domaine borné régulier et soumis aux conditions aux limites de Dirichlet (4).

☆ Support by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, is acknowledged.
E-mail addresses: naoufel@mip.ups-tlse.fr (N. Ben Abdallah), mehats@mip.ups-tlse.fr (F. Méhats), vauchel@mip.ups-tlse.fr (N. Vauchelet).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.09.025

Si les données au bord sont à l'équilibre thermodynamique, c'est-à-dire s'il existe un réel $u_\infty > 0$ tel que $N_b = u_\infty e^{-V_b}$ pour tout $x \in \partial\Omega$, alors le problème stationnaire associé (5) admet une unique solution (N_∞, V_∞), appelée *solution d'équilibre*. Dans un premier temps, en supposant l'existence de solutions pour (2)–(4) établie pour tout temps, nous intéressons à la convergence en temps long de ces solutions vers la solution d'équilibre. Le résultat principal de cette note est le suivant :

Théorème 0.1. *Soit une solution* (N, V) *au problème* (2)–(4) *satisfaisant* $N \in C(\mathbb{R}_+, L^2(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega))$ *et* $V \in C(\mathbb{R}_+, H^1(\Omega))$. *Si les conditions aux bords sont à l'équilibre thermodynamique alors il existe* $C > 0$ *et* $\lambda > 0$ *tels que pour tout* $t \geqslant 0$

$$\|N - N_\infty\|_{L^2(\Omega)}(t) + \|V - V_\infty\|_{H^1(\Omega)}(t) \leqslant Ce^{-\lambda t}. \tag{1}$$

Pour démontrer ce résultat, nous utilisons l'entropie relative entre (N, V) et (N_∞, V_∞) définie par (7). Nous établissons dans un premier temps la décroissance et la convergence en temps long de cette entropie relative vers 0, ce qui permet de montrer la convergence de N vers N_∞ dans $L^1(\Omega)$ et de V vers V_∞ dans $H^1(\Omega)$. Ensuite, nous considérons un développement quadratique L de l'entropie relative autour de N_∞ et V_∞, donné par (11). Pour prouver (1), il suffit de démontrer que cette quantité L converge exponentiellement vers 0. L'inégalité cruciale qui le permet est (14).

Dans un second temps, nous intéressons à l'existence pour le système (2)–(4) de solutions satisfaisant les hypothèses du Théorème précédent. En réalité, nous construisons des solutions dans un cadre $L \log L$, la densité n'appartenant à $L^2(\Omega)$ qu'à partir d'un certain temps :

Théorème 0.2. *Si les conditions aux bords sont à l'équilibre thermodynamique, alors le problème* (2)–(4) *admet une solution faible* (N, V) *vérifiant* $V \in C(\mathbb{R}_+, H^1(\Omega))$ *et* $N \in C(\mathbb{R}_+, L^1(\Omega))$. *De plus il existe* $T > 0$ *tel que* $N \in C([T, \infty), L^2(\Omega))$ *et il existe* $C > 0$ *et* $\lambda > 0$ *tels que cette solution vérifie* (1) *pour* $t \geqslant T$.

1. Introduction

This note is devoted to the analysis of the long time behavior of solutions of a drift-diffusion-Poisson system arising from plasmas, semiconductors or electrolytes physics. The transport of carriers is described by a drift-diffusion equation governing the evolution of the density of particles $N(t, x)$. This equation is coupled to the Poisson equation, satisfied by the selfconsistent electrostatic potential $V(t, x)$. Let $\Omega \subset \mathbb{R}^3$ be a bounded regular domain. We consider the following coupled system:

$$\partial_t N - \operatorname{div}\big(\mathbb{D}\,(\nabla N + N\,\nabla V)\big) = 0, \tag{2}$$

$$-\Delta V = N. \tag{3}$$

In this system, $\mathbb{D} = \mathbb{D}(x)$ is the diffusion matrix, smooth enough and uniformly positive definite:

Assumption 1.1. The function $\mathbb{D}$ is a C^1 function on $\overline{\Omega}$ into the set of 2×2 symmetric positive definite matrix such that for all $x \in \Omega$ we have $\mathbb{D}(x) \geqslant \alpha I$, where $\alpha > 0$ is given.

This system (2), (3) is supplemented with the initial condition $N(0, x) = N^0(x)$ and with Dirichlet conditions on the boundary $\partial\Omega$ of Ω:

$$N = N_b \quad \text{and} \quad V = V_b \quad \text{on } \partial\Omega. \tag{4}$$

These Cauchy and Dirichlet data satisfy the following assumption:

Assumption 1.2. The Cauchy data N^0 is nonnegative and belongs to $L^{6/5}(\Omega)$. Moreover, we have $V_b \in C^2(\partial\Omega)$ and we assume that there exists a real number $u_\infty > 0$ such that $N_b = u_\infty e^{-V_b}$.

Here the major assumption is that the boundary data are at *thermal equilibrium*.

Entropy methods are now classical tools for the analysis of the long time behavior of diffusive transport models. In 1985, Gajewski [9] used the relative entropy to construct solutions and prove their convergence to the equilibrium for drift-diffusion-Poisson systems in bounded domains (with general mixed boundary conditions), in the case of scalar diffusion ($\mathbb{D} = \alpha I$). This work was followed by [10], where this convergence was precised and shown to be exponential. More recently, the whole-space case with confining potential [2], its generalization to bipolar systems [3], the case of bounded domains with no-flux boundary conditions [6] as well as systems with nonlinear diffusion [7] were treated by entropy methods. One can also refer to [11] for an application to the analysis of the quasineutral limit. All these recent works, based on logarithmic Sobolev inequalities, use strongly the fact that system is isolated (the L^1 norm of N is conserved). In our case, due to the ohmic contacts, this property is not true. Nevertheless, we propose here a method based on a quadratic relative entropy which enables to prove the exponential convergence for (2), (3) subject to Dirichlet boundary conditions (4) and with a diffusion matrix. This idea is not new and has been used in linear situation for analyzing the spectral gap of the operator in (2) (see, e.g., [2] and references therein). We also applied this technique in [5] to study the long time behavior of solutions of a drift-diffusion equation coupled to a Schrödinger–Poisson system.

In order to present our results, let us introduce the stationary drift-diffusion-Poisson problem. This system, of unknowns (N_∞, V_∞), reads

$$-\operatorname{div}\big(\mathbb{D}(\nabla N_\infty + N_\infty \nabla V_\infty)\big) = 0, \qquad -\Delta V_\infty = N_\infty \tag{5}$$

subject to the boundary conditions: $N_\infty = N_b$ and $V_\infty = V_b$ on $\partial\Omega$. Under Assumption 1.2, this system (5) admits a unique solution, which is written $N_\infty = u_\infty e^{-V_\infty}$, where $V_\infty \in C^2(\overline{\Omega})$, and can be defined as the minimizer of a convex functional [8]. Our main result is

Theorem 1.3. *If Assumptions* 1.1 *and* 1.2 *hold and, if* (N, V) *is a solution of* (2)–(4) *such that* $N \in C(\mathbb{R}_+, L^2(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega))$, $V \in C(\mathbb{R}_+, H^1(\Omega))$, *then there exist two constants* $C > 0$ *and* $\lambda > 0$ *such that, for any* $t \geqslant 0$,

$$\|N - N_\infty\|_{L^2(\Omega)}(t) + \|V - V_\infty\|_{H^1(\Omega)}(t) \leqslant C e^{-\lambda t}. \tag{6}$$

Remark that this theorem implicitly assumes that the initial data belongs to $L^2(\Omega)$ (which is stronger than Assumption 1.2). This theorem will be proved in two steps. In Section 2, we introduce the relative entropy of (N, V) to (N_∞, V_∞) and explain how the decay of this quantity gives the convergence to the equilibrium as $t \to +\infty$. In Section 3, we prove the exponential convergence stated in (6). In Section 4 we investigate the existence of solutions (N, V) which satisfy the assumptions of Theorem 1.3. In fact, due to the lack of estimates, we only prove the existence of solutions which satisfy these assumptions after a certain time. Namely, in this section we sketch the proof of:

Theorem 1.4. *Under Assumptions* 1.1 *and* 1.2, *there exists a weak solution* (N, V) *of* (2)–(4) *which satisfies* $N \in C(\mathbb{R}_+, L^1(\Omega))$, $V \in C(\mathbb{R}_+, H^1(\Omega))$. *Moreover there exists* $T > 0$ *such that* $N \in C([T, \infty), L^2(\Omega))$ *and there exist* $C > 0$ *and* $\lambda > 0$ *such that this solution satisfies* (6) *for* $t \geqslant T$.

2. Convergence of the relative entropy

The relative entropy of (N, V) to (N_∞, V_∞) being defined by

$$E(t) = \int_\Omega \left(N \log \frac{N}{N_\infty} + N_\infty - N \right) \mathrm{d}x + \frac{1}{2} \int_\Omega \big|\nabla (V - V_\infty)\big|^2 \,\mathrm{d}x, \tag{7}$$

the following lemma states that this quantity decreases towards 0 as t goes to $+\infty$:

Lemma 2.1. *Let* (N, V) *satisfy* (2), (3) *in the sense precised in Theorem* 1.3 *and let* (N_∞, V_∞) *satisfy* (5). *Then the relative entropy defined by* (7) *is decreasing to* 0 *as* t *goes to* $+\infty$. *This implies that* N *and* V *converge to* N_∞ *and* V_∞ *respectively in* $L^1(\Omega)$ *and* $H^1(\Omega)$.

Proof. Straightforward calculations using (2)–(4) and Assumptions 1.1, 1.2 give $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_\Omega e^{-V}\frac{\mathbb{D}\nabla u\cdot\nabla u}{u}\,\mathrm{d}x \leqslant -\alpha\int_\Omega e^{-V}\frac{|\nabla u|^2}{u}\,\mathrm{d}x$, where $u = Ne^V$. Hence the entropy is a decreasing function of time and for all $t \geqslant 0$ we have

$$E(t) + \alpha\int_0^t\int_\Omega e^{-V}\frac{|\nabla u|^2}{u}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant E(0). \tag{8}$$

This inequality is enough to prove the lemma, thanks to Gajewski's argument [9]. For the sake of clarity of this note, let us briefly summarize this argument. First, there exists a sequence $t_j \to +\infty$ such that

$$\lim_{j\to+\infty}\int_\Omega e^{-V(t_j)}\frac{|\nabla u(t_j)|^2}{u(t_j)}\,\mathrm{d}x = 0. \tag{9}$$

Besides, we deduce from direct calculations that $\int_\Omega e^{-V}\frac{|\nabla u|^2}{u}\,\mathrm{d}x = \int_\Omega 4|\nabla\sqrt{N}\,|^2\,\mathrm{d}x + \int_\Omega N|\nabla V|^2\,\mathrm{d}x + \int_\Omega 2N^2\,\mathrm{d}x + \int_{\partial\Omega} N_b\,\partial_\nu V\,\mathrm{d}s$ (where $\nu(x)$ is the outward normal vector at $x \in \partial\Omega$ and $\mathrm{d}s$ is the surface measure). Since, by (3) and the standard properties of elliptic equations, we have $|\int_{\partial\Omega} N_b\partial_\nu V\,\mathrm{d}s| \leqslant C\|V\|_{H^2(\Omega)} \leqslant C(\|N\|_{L^2(\Omega)} + \|V_b\|_{H^{3/2}(\partial\Omega)}) \leqslant C + \int_\Omega N^2\,\mathrm{d}x$, we deduce that

$$\int_\Omega e^{-V}\frac{|\nabla u|^2}{u}\,\mathrm{d}x \geqslant \int_\Omega 4\left|\nabla\sqrt{N}\,\right|^2\mathrm{d}x - C_0. \tag{10}$$

Then (9) and (10) imply that $\|\sqrt{N(t_j)}\|_{H^1(\Omega)} \leqslant C$. Up to an extraction of subsequence, we can prove that $\lim_{j\to+\infty} E(t_j) = 0$. Since E is decreasing, $\lim_{t\to+\infty} E(t) = 0$. Applying Csiszár–Kullback inequality [1], we deduce that $\|N - N_\infty\|_{L^1(\Omega)}(t) \to 0$ and $\|V - V_\infty\|_{H^1(\Omega)}(t) \to 0$, as $t \to +\infty$. □

3. Exponential convergence

This section is devoted to the proof of Theorem 1.3. Setting $n = N - N_\infty$ and $v = V - V_\infty$, the lowest order term of an expansion of $E(t)$ is

$$L(t) = \frac{1}{2}\int_\Omega \frac{n^2}{N_\infty}\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x. \tag{11}$$

From (2), (3), (11) and after an integration by parts, it comes

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\int_\Omega N_\infty\mathbb{D}\nabla\left(\frac{n}{N_\infty} + v\right)\cdot\nabla\left(\frac{n}{N_\infty} + v\right)\mathrm{d}x - \int_\Omega n\mathbb{D}\nabla v\cdot\nabla\left(\frac{n}{N_\infty} + v\right)\mathrm{d}x. \tag{12}$$

By the Cauchy–Schwarz inequality we have $-\int_\Omega n\mathbb{D}\nabla v\cdot\nabla(\frac{n}{N_\infty} + v)\,\mathrm{d}x \leqslant \frac{1}{2}\int_\Omega N_\infty\mathbb{D}\nabla(\frac{n}{N_\infty} + v)\cdot\nabla(\frac{n}{N_\infty} + v)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{n^2}{N_\infty}\mathbb{D}\nabla v\cdot\nabla v\,\mathrm{d}x$. Then by using Assumption 1.1 and the fact that N_∞ is bounded from below, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leqslant -C_1\int_\Omega\left(\left|\nabla\left(\frac{n}{N_\infty}\right)\right|^2 + 2\frac{n^2}{N_\infty} + |\nabla v|^2\right)\mathrm{d}x + C_2\int_\Omega n^2|\nabla v|^2\,\mathrm{d}x. \tag{13}$$

Besides, by Gagliardo–Nirenberg and Sobolev inequalities and by using the Poisson equation to estimate the $L^6(\Omega)$ norm of ∇v in terms of $\|n\|_{L^2}$, we obtain $\int_\Omega n^2|\nabla v|^2\,\mathrm{d}x \leqslant C\|\nabla v\|_{L^2(\Omega)}^{1/2}\|n\|_{L^2(\Omega)}^2\|\frac{n}{N_\infty}\|_{H^1(\Omega)}^{3/2} \leqslant$

$C_3\varepsilon\|\nabla(\frac{n}{N_\infty})\|^2_{L^2(\Omega)} + \frac{C_4}{\varepsilon^3}\|n\|^8_{L^2(\Omega)}$, where we also used the fact that ∇v is bounded in $L^\infty(\mathbb{R}_+, L^2(\Omega))$ (see (8)) and the Poincaré inequality. Hence, by fixing $\varepsilon > 0$ small enough such that $C_2 C_3 \varepsilon < C_1$, Eq. (13) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leqslant -C_1 \int_\Omega \left(2\frac{n^2}{N_\infty} + |\nabla v|^2\right)\mathrm{d}x + \frac{C_2 C_4}{\varepsilon^3}\|n\|^8_{L^2(\Omega)} \leqslant -2C_1 L(t) + C_5 L(t)^4. \tag{14}$$

We have seen in the proof of Lemma 2.1 that there exists a sequence $(t_j)_{j\in\mathbb{N}}$ going to $+\infty$ such that $(\sqrt{N(t_j)})_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Thanks to the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, $\|n(t_j)\|_{L^3(\Omega)}$ is bounded for all $j \in \mathbb{N}$. Moreover Lemma 2.1 implies that $\lim_{j\to+\infty}\|n(t_j)\|_{L^1(\Omega)} = 0$. Hence, by interpolation, we deduce that $\lim_{j\to+\infty} L(t_j) = 0$. Consequently, one can find t_* such that $C_5 L(t_*)^3 \leqslant C_1$. By (14), we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}L(t_*) \leqslant -C_1\, L(t_*) < 0. \tag{15}$$

Finally, a continuation argument shows that (15) holds for any $t \geqslant t_*$.

4. About existence of solutions

In this section, we prove Theorem 1.4. The natural estimate for the system (2), (3) is the entropy estimate (8), but the associated $L\log L$ space for N is not sufficient to justify the calculations done above. Moreover, due to the matrix $\mathbb{D}$, the standard L^2 estimate for the drift-diffusion-Poisson system cannot be obtained as usual by a simple multiplication of (2) by N. Hence, let us regularize the problem. A classical technique (see, e.g., [4,13]) consists in considering, for $\varepsilon > 0$ and m large enough ($m \geqslant 2$ suffices), the modified problem:

$$\partial_t N^\varepsilon - \operatorname{div}\big(\mathbb{D}\big(\nabla N^\varepsilon + N^\varepsilon\,\nabla V^\varepsilon\big)\big) = 0, \tag{16}$$

$$-(1-\varepsilon\Delta)^{2m}\Delta V^\varepsilon = N^\varepsilon \tag{17}$$

subject to a regularized initial condition $N(0,x) = N^{0,\varepsilon}(x)$ and the boundary conditions:

$$N^\varepsilon(t,x) = N_b, \quad V^\varepsilon(t,x) = V_b \quad \text{and} \quad \Delta V^\varepsilon = \cdots = \Delta^{2m}V^\varepsilon = 0, \quad \text{for } x \in \partial\Omega. \tag{18}$$

We also define N^ε_∞, V^ε_∞ as the solution of the corresponding stationary system. By classical arguments based on a fixed point theorem (see [12]), one can show that this problem admits a solution $(N^\varepsilon, V^\varepsilon)$ which satisfies $N^\varepsilon \in C(\mathbb{R}_+, L^2(\Omega))$ and $V^\varepsilon \in C(\mathbb{R}_+, H^{4m+2}(\Omega))$. For this solution, we follow the strategy developed in Sections 2 and 3. Let E^ε be the relative entropy of the new system (16), (17):

$$E^\varepsilon(t) = \int_\Omega \left(N^\varepsilon \log\frac{N^\varepsilon}{N^\varepsilon_\infty} + N^\varepsilon_\infty - N^\varepsilon\right)\mathrm{d}x + \frac{1}{2}\int_\Omega \big|(1-\varepsilon\Delta)^m\nabla\big(V^\varepsilon - V^\varepsilon_\infty\big)\big|^2\,\mathrm{d}x. \tag{19}$$

One can check easily that, for all $t > 0$,

$$E^\varepsilon(t) + \alpha\int_0^t\int_\Omega e^{-V^\varepsilon}\frac{|\nabla u^\varepsilon|^2}{u^\varepsilon}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant E^\varepsilon(0) \leqslant C, \quad \text{where } u^\varepsilon = N^\varepsilon e^{V^\varepsilon} \tag{20}$$

(this constant C can be chosen independent of ε thanks to the assumption $N^0 \in L^{6/5}(\Omega)$). Since we have $\|\Delta V^\varepsilon\|_{L^2(\Omega)} \leqslant \|(1-\varepsilon\Delta)^m\Delta V^\varepsilon\|_{L^2(\Omega)}$ (Lemma 3.4 of [4]), we can show as before that

$$\int_\Omega e^{-V^\varepsilon}\frac{|\nabla u^\varepsilon|^2}{u^\varepsilon}\,\mathrm{d}x \geqslant \int_\Omega 4\big|\nabla\sqrt{N^\varepsilon}\,\big|^2\,\mathrm{d}x - C. \tag{21}$$

It is readily seen that these estimates (19) and (21) enable to pass to the limit as ε goes to 0. Indeed, it is clear that $\sqrt{N^\varepsilon}$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega)) \cap L^\infty(\mathbb{R}_+, L^2(\Omega))$ independently of ε. Hence, by using Sobolev embeddings and interpolation estimates, we obtain an estimate for N^ε in $L^{8/7}_{\mathrm{loc}}(\mathbb{R}_+, W^{1,4/3}(\Omega))$. Since ∇V^ε is bounded in $L^\infty(\mathbb{R}_+, L^2(\Omega))$, one can also get a bound of $\partial_t N^\varepsilon$ directly from (16). To pass to the limit, it suffices then to apply an Aubin–Lions compactness lemma.

As in Section 2, one can prove thanks to (19) and (21) that the entropy E^ε decreases to 0 as t goes to $+\infty$. Let us now consider the following quantity (with the notations n^ε, v^ε defined as in Section 3):

$$L^\varepsilon(t) = \frac{1}{2}\int_\Omega \frac{(n^\varepsilon)^2}{N^\varepsilon_\infty}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \left|\nabla v^\varepsilon\right|^2 \mathrm{d}x. \tag{22}$$

Similar calculations lead to inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}L^\varepsilon(t) \leqslant -2C_1 L^\varepsilon(t) + C_5 L^\varepsilon(t)^4, \tag{23}$$

with C_1 and C_5 independent of ε. In Section 3 we have proved that if there exists $t^\varepsilon_* > 0$ such that $L^\varepsilon(t^\varepsilon_*)^3 \leqslant C_1/C_5$, then there exists two positive constants C and λ independent of ε such that for all $t \geqslant t^\varepsilon_*$, $L^\varepsilon(t) \leqslant Ce^{-\lambda t}$. The crucial point is to find such t^ε_* bounded uniformly with respect to ε.

First, we can prove easily by a contradiction argument that for all $\eta > 0$ there exists $\alpha(\eta) > 0$ such that,

$$\left(\forall\varepsilon > 0,\ \forall t \geqslant 0,\ \int_\Omega e^{-V^\varepsilon(t)}\frac{|\nabla u^\varepsilon(t)|^2}{u^\varepsilon(t)}\,\mathrm{d}x \leqslant \alpha(\eta)\right) \quad\Rightarrow\quad L^\varepsilon(t) \leqslant \eta. \tag{24}$$

Next, inequality (20) implies that there exists a nonnegative constant C_0 such that

$$\forall T > 0,\ \exists t^\varepsilon(T) \in [0, T] \quad \text{such that} \quad \int_\Omega e^{-V^\varepsilon(t^\varepsilon(T))}\frac{|\nabla u^\varepsilon(t^\varepsilon(T))|^2}{u^\varepsilon(t^\varepsilon(T))}\,\mathrm{d}x \leqslant \frac{C_0}{T}. \tag{25}$$

Thus, choosing $\eta = (C_1/C_5)^{1/3}$ in (24) and $T \geqslant C_0/\alpha(\eta)$ in (25), we deduce that the corresponding $t^\varepsilon(T)$ can be chosen as t^ε_*.

Finally, on $[T, \infty)$ we have the estimate $L^\varepsilon(t) \leqslant Ce^{-\lambda t}$. Since for all $t \geqslant T$ we have $L(t) \leqslant \liminf_{\varepsilon\to 0} L^\varepsilon(t) \leqslant Ce^{-\lambda t}$ (L defined in (11)), we deduce that N belongs to $C([T, \infty), L^2(\Omega))$.

References

[1] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On generalized Csiszár–Kullback inequalities, Monatsh. Math. 131 (3) (2000) 235–253.

[2] A. Arnold, P.A. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differential Equations 26 (1–2) (2001) 43–100.

[3] A. Arnold, P. Markowich, G. Toscani, On large time asymptotics for drift-diffusion-Poisson systems, Transport Theory Statist. Phys. 29 (3–5) (2000) 571–581.

[4] N. Ben Abdallah, Weak solutions of the initial-boundary value problem for the Vlasov–Poisson system, Math. Methods Appl. Sci. 17 (6) (1994) 451–476.

[5] N. Ben Abdallah, F. Méhats, N. Vauchelet, Diffusive transport of partially quantized particle: existence, uniqueness and long time behavior, submitted for publication.

[6] P. Biler, J. Dolbeault, Long time behavior of solutions of Nernst–Planck and Debye–Hückel drift-diffusion systems, Ann. Henri Poincaré 1 (3) (2000) 461–472.

[7] P. Biler, J. Dolbeault, P.A. Markowich, Large time asymptotics of nonlinear drift-diffusion systems with Poisson coupling, Transport Theory Statist. Phys. 30 (4–6) (2001) 521–536.

[8] J. Dolbeault, Stationary states in plasma physics: Maxwellian solutions of the Vlasov–Poisson system, Math. Models Methods Appl. Sci. 1 (2) (1991) 183–208.

[9] H. Gajewski, On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors, Z. Angew. Math. Mech. 65 (2) (1985) 101–108.

[10] H. Gajewski, K. Gröger, Semiconductor equations for variable mobilities based on Boltzmann statistics or Fermi–Dirac statistics, Math. Nachr. 140 (1989) 7–36.

[11] I. Gasser, C.D. Levermore, P.A. Markowich, C. Schmeiser, The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model, European J. Appl. Math. 12 (4) (2001) 497–512.

[12] P.A. Markowich, C.A. Ringhofer, C. Schmeiser, Semiconductor Equations, Springer-Verlag, Vienna, 1990.

[13] F. Poupaud, Boundary value problems for the stationary Vlasov–Maxwell systems, Forum Math. 4 (1992) 499–527.