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Partial Differential Equations

On the asymptotic behaviour of elliptic problems with periodic data

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Abstract

We study the asymptotic behaviour of the solution of elliptic problems with periodic data when the size of the domain on which the problem is set becomes unbounded. ***To cite this article: M. Chipot, Y. Xie, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur le comportement asymptotique de problèmes elliptiques à données périodiques. On s'intéresse au comportement asymptotique de la solution de problèmes elliptiques à données périodiques lorsque la taille de l'ouvert sur lequel le problème est posé devient infinie. ***Pour citer cet article : M. Chipot, Y. Xie, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit T un nombre positif. Pour $n, k \geqslant 1$ entiers on pose

$$\Omega_n = (-nT, nT)^k, \qquad Q = (0, T)^k.$$

On dit que $f \in L^1_{\mathrm{Loc}}(\mathbb{R}^k)$ est T-périodique dans toutes les directions si l'on a

$$f(x + Te_j) = f(x) \quad \text{p.p. } x \in \mathbb{R}^k, \quad \forall j = 1, \dots, k.$$

((e_j) désigne la base canonique de $\mathbb{R}^k$). On considère alors a_{ij}, $i, j = 1, \dots, k$, a, f_i, $i = 0, \dots, k$, des fonctions T-périodiques dans toutes les directions. Soit V_n un sous espace de $H^1(\Omega_n)$ tel que

$$V_n \text{ est fermé dans } H^1(\Omega_n), \qquad H^1_0(\Omega_n) \subset V_n \subset H^1(\Omega_n).$$

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On désigne par u_n la solution faible de

$$\begin{cases} u_n \in V_n, \\ \displaystyle\int_{\Omega_n} a_{ij}(x)\partial x_i u_n \partial x_j v + a(x) u_n v \,\mathrm{d}x = \int_{\Omega_n} f_0 v + f_i \partial x_i v \,\mathrm{d}x \quad \forall v \in V_n, \end{cases}$$

et par u_∞ la solution de

$$\begin{cases} u_\infty \in H^1_{\mathrm{per}}(Q), \\ \displaystyle\int_{Q} a_{ij}(x)\partial x_i u_\infty \partial x_j v + a(x) u_\infty v \,\mathrm{d}x = \int_{Q} f_0 v + f_i \partial x_i v \,\mathrm{d}x \quad \forall v \in H^1_{\mathrm{per}}(Q), \end{cases}$$

où $H^1_{\mathrm{per}}(Q)$ est defini par

$$H^1_{\mathrm{per}}(Q) = \{v \in H^1(Q) \mid v \text{ est } T\text{-périodique dans toutes les directions}\}.$$

Sous les hypothèses usuelles d'ellipticité on se propose de montrer que

– lorsque $0 \leqslant a(x) \leqslant \Lambda$, $a \not\equiv 0$, pour tout $n_0 > 0$ et tout $r > 0$, il existe une constante C indépendante de n telle que

$$|u_n - u_\infty|_{H^1(\Omega_{n_0})} \leqslant \frac{C}{n^r}.$$

(Par souci de simplicité nous donnerons ici la preuve de ce résultat dans le cas où $0 < \lambda \leqslant a(x)$ renvoyant le lecteur à [2] pour le cas général.)

– Lorsque $a \equiv 0$, $\int_Q f_0 \,\mathrm{d}x = 0$, $V_n = H^1_0(\Omega_n)$, et si u_∞ est solution de (7) de moyenne nulle, alors il existe une constante C telle que (à une sous suite près)

$$u_n \rightharpoonup u_\infty + C \quad \text{dans } L^\infty(\mathbb{R}^k) \text{ faible } *.$$

Autrement dit, dans presque tous les cas (sauf quand $a \equiv 0$, $\int_Q f_0 \not\equiv 0$ où u_n est non borné), les données périodiques forcent la convergence sur tout compact vers une fonction périodique.

1. Introduction

Let T denote a positive number. For n, k integers $n, k \geqslant 1$ we set

$$\Omega_n = (-nT, nT)^k, \qquad Q = (0, T)^k. \tag{1}$$

We say that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^k)$ is T-periodic in all directions if it holds that

$$f(x + Te_j) = f(x) \qquad \text{a.e. } x \in \mathbb{R}^k,\ \forall j = 1, \ldots, k. \tag{2}$$

((e_j) denotes the canonical basis of $\mathbb{R}^k$). Consider then

$$a_{ij},\ i, j = 1, \ldots, k,\ a, \qquad \text{functions in } L^\infty(\mathbb{R}^k),\ T\text{-periodic in all directions}, \tag{3}$$

$$f_i,\ i = 0, \ldots, k, \qquad \text{functions in } L^2(Q),\ T\text{-periodic in all directions}. \tag{4}$$

(We suppose of course that all the functions we use are defined in all $\mathbb{R}^k$ – extended eventually by periodicity.) Let us denote by V_n a subspace of $H^1(\Omega_n)$ such that

$$V_n \text{ is closed in } H^1(\Omega_n), \qquad H^1_0(\Omega_n) \subset V_n \subset H^1(\Omega_n). \tag{5}$$

(We refer the reader to [5,1] for the notation used in this Note.) We denote by u_n the weak solution to

$$\begin{cases} u_n \in V_n, \\ \int_{\Omega_n} a_{ij}(x)\partial x_i u_n \partial x_j v + a(x)u_n v\,\mathrm{d}x = \int_{\Omega_n} f_0 v + f_i \partial x_i v\,\mathrm{d}x \quad \forall v \in V_n, \end{cases} \tag{6}$$

and by u_∞ the solution to

$$\begin{cases} u_\infty \in H^1_{\mathrm{per}}(Q), \\ \int_Q a_{ij}(x)\partial x_i u_\infty \partial x_j v + a(x)u_\infty v\,\mathrm{d}x = \int_Q f_0 v + f_i \partial x_i v\,\mathrm{d}x \quad \forall v \in H^1_{\mathrm{per}}(Q), \end{cases} \tag{7}$$

where $H^1_{\mathrm{per}}(Q)$ is defined by

$$H^1_{\mathrm{per}}(Q) = \{v \in H^1(Q) \mid v \text{ is } T\text{-periodic in all directions}\}. \tag{8}$$

Under the usual ellipticity conditions described below it is clear that when $0 \leqslant a$, $a \not\equiv 0$ then both (6), (7) admit a unique solution. Then we would like to show that u_n converges towards u_∞.

In the case where $a \equiv 0$, that we call the degenerate case, (6) possesses a unique solution when $V_n = H^1_0(\Omega_n)$. Moreover, if we impose to u_∞ to be of average of 0 then (7) admits also a unique solution. Then, roughly speaking, when f_0 is of average 0, we will show that up to a constant u_n converges towards u_∞.

2. The nondegenerate case

In this section we assume that for some positive constant λ, Λ it holds that

$$0 < \lambda \leqslant a(x) \leqslant \Lambda \quad \text{a.e. } x \in \mathbb{R}^k, \tag{9}$$

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \quad \text{a.e. } x \in \mathbb{R}^k,\ \forall \xi \in \mathbb{R}^k. \tag{10}$$

((10) is the usual ellipticity condition – there is no loss of generality in assuming λ to be the same in (9) and (10)). Clearly, under the above assumptions there exists a unique solution to (6) and also to (7). Moreover we have:

Theorem 2.1. *For any $n_0 > 0$ and any exponent $r > 0$ there exists a constant C independent of n such that*

$$|u_n - u_\infty|_{H^1(\Omega_{n_0})} \leqslant \frac{C}{n^r}. \tag{11}$$

In the above estimate – as in the following – we set

$$|u|_{H^1(\Omega)} = \left[\int_\Omega \left\{|\nabla u|^2 + u^2\right\}\mathrm{d}x\right]^{1/2}. \tag{12}$$

In order to prove our theorem, we will need some lemmas. First

Lemma 2.2 (Estimate of u_n). *It holds that*

$$|u_n|^2_{H^1(\Omega_n)} \leqslant \frac{2^k}{\lambda^2}|f|^2_{2,Q} n^k \tag{13}$$

where we have set

$$|f|^2_{2,Q} = \int_Q |f|^2\,\mathrm{d}x = \int_Q \sum_{i=0}^k f_i^2\,\mathrm{d}x. \tag{14}$$

Proof of the Lemma. We take $v = u_n$ as test function in (6). Using (9), (10) and the Cauchy–Schwarz inequality it comes after some easy computations

$$\lambda |u_n|^2_{H^1(\Omega_n)} \leqslant \int_{\Omega_n} a_{ij}(x)\partial x_i u_n \partial x_j u_n + a u_n^2 \,\mathrm{d}x = \int_{\Omega_n} f_0 u_n + f_i \partial x_i u_n \,\mathrm{d}x \leqslant \left(\int_{\Omega_n} |f|^2 \,\mathrm{d}x \right)^{1/2} |u_n|_{H^1(\Omega_n)}$$

($|f|^2 = \sum_{1=0}^k f_i^k$). (14) follows easily due to the periodicity of $f = (f_0, \ldots, f_k)$. □

Next we will use the following result well known in homogenization theory (see [4] for an idea of the proof).

Lemma 2.3. *The solution u_∞ of* (7) *satisfies*

$$-\partial x_j\big(a_{ij}(x)\partial x_i u_\infty\big) + a(x)u_\infty = f_0 - \partial x_i f_i \quad \textit{in } \mathcal{D}'(\mathbb{R}^k). \tag{15}$$

Proof of Theorem 2.1. Let ϱ be the piecewise continuous affine function such that

$$\varrho = 1 \quad \text{on } \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \varrho = 0 \text{ outside } (-1, 1), \quad \varrho \text{ is affine on } \left(-1, -\frac{1}{2}\right), \left(\frac{1}{2}, 1\right). \tag{16}$$

It is clear that it holds that

$$|\varrho'| \leqslant 2. \tag{17}$$

Moreover, for any $n_1 \leqslant n$

$$(u_n - u_\infty) \prod_{i=1}^k \varrho^2\left(\frac{x_i}{n_1 T}\right) := (u_n - u_\infty)\Pi^2 \tag{18}$$

is a test function for (6). It follows that it holds – see Lemma 2.3

$$\int_{\Omega_{n_1}} a_{ij}\partial x_i(u_n - u_\infty)\partial x_j\big\{(u_n - u_\infty)\Pi^2\big\} + a(u_n - u_\infty)^2\Pi^2 \,\mathrm{d}x = 0. \tag{19}$$

This leads to

$$\int_{\Omega_{n_1}} a_{ij}\partial x_i(u_n - u_\infty)\partial x_j(u_n - u_\infty)\Pi^2 + a(u_n - u_\infty)^2\Pi^2 \,\mathrm{d}x = -2\int_{\Omega_{n_1}} a_{ij}\partial x_i(u_n - u_\infty)\partial x_j \Pi (u_n - u_\infty)\Pi.$$

Since $\partial x_j \Pi = \frac{1}{n_1 T}\varrho'(\frac{x_j}{n_1 T})\prod_{i\neq j}\varrho(\frac{x_i}{n_1 T})$ we derive easily

$$\lambda \int_{\Omega_{n_1}} \big\{\big|\nabla(u_n - u_\infty)\big|^2 + (u_n - u_\infty)^2\big\}\Pi^2 \,\mathrm{d}x \leqslant \frac{C}{n_1}\int_{\Omega_{n_1}} \big|\nabla(u_n - u_\infty)\big||u_n - u_\infty|\Pi \,\mathrm{d}x$$

where $C = C(T, a_{ij})$ is independent of n. Using Cauchy–Schwarz inequality we obtain

$$\lambda \int_{\Omega_{n_1}} \{|\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2\}\Pi^2 \,\mathrm{d}x \leqslant \frac{C}{n_1}\left\{\int_{\Omega_{n_1}} \big|\nabla(u_n - u_\infty)\big|^2\Pi^2 \,\mathrm{d}x\right\}^{1/2}\left\{\int_{\Omega_{n_1}} (u_n - u_\infty)^2 \,\mathrm{d}x\right\}^{1/2}.$$

Thus – for some constant C independent of n –

$$\int_{\Omega_{n_1}} \big\{\big|\nabla(u_n - u_\infty)\big|^2 + (u_n - u_\infty)^2\big\}\Pi^2 \,\mathrm{d}x \leqslant \frac{C}{n_1^2}\int_{\Omega_{n_1}} \big|\nabla(u_n - u_\infty)\big|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x.$$

Due to the definition of Π this implies

$$|u_n - u_\infty|_{H^1(\Omega_{n_1/2})} \leqslant \frac{C}{n_1}|u_n - u_\infty|_{H^1(\Omega_{n_1})} \quad \forall n_1 \leqslant n.$$

Choosing $n_1 = \frac{n}{2^{p-1}}$ and iterating the above formula leads to

$$|u_n - u_\infty|_{H^1(\Omega_{n/2^p})} \leqslant \frac{C}{n^p}|u_n - u_\infty|_{H^1(\Omega_n)} \leqslant \frac{C}{n^{p-k/2}}$$

(cf. Lemma 2.2). Choosing $\frac{n}{2^p} \geqslant n_0$, $p - \frac{k}{2} > r$ the result follows. □

Remark 1. The method allows us to deal with more general periodic data (cf. [2]) also the parabolic analogue could be considered, as well as nonlinear versions – cf. [3,1]. We can extend the results to more general Ω_n than (1). Note also that our convergence result does not depend on our boundary conditions on $\partial\Omega_n$. It is also valid when (9) is replaced by

$$0 \leqslant a(x) \leqslant \Lambda \quad \text{a.e. } x \in \mathbb{R}^k, \quad a \not\equiv 0, \tag{20}$$

(see [2] for details).

3. The degenerate case

In this section we assume that

$$a \equiv 0. \tag{21}$$

Moreover we consider only the case of homogeneous boundary conditions that is to say the case

$$V_n = H_0^1(\Omega_n). \tag{22}$$

Then it is easy to see that, in order to prevent u_n solution of (6) to be unbounded, one has to assume

$$\int_Q f_0(x)\,\mathrm{d}x = 0. \tag{23}$$

Under this assumption the second hand side of (7) defines a continuous linear form on

$$\overline{H}^1_{\text{per}}(Q) = \left\{ v \in H^1_{\text{per}}(Q) \;\middle|\; \int_Q v\,\mathrm{d}x = 0 \right\} \tag{24}$$

and by the Lax–Milgram theorem there exists a unique u_∞ solution to

$$\begin{cases} u_\infty \in \overline{H}^1_{\text{per}}(Q), \\ \displaystyle\int_Q a_{ij}(x)\partial_{x_i}u_\infty\partial_{x_j}v\,\mathrm{d}x = \int_Q f_0 v + f_i\partial_{x_i}v\,\mathrm{d}x \quad \forall v \in \overline{H}^1_{\text{per}}(Q). \end{cases} \tag{25}$$

Moreover we have:

Theorem 3.1. *Under the above assumptions, if in addition*

$$u_\infty \in L^\infty(Q), \tag{26}$$

there exists a subsequence of u_n that we still label by n such that

$$u_n \rightharpoonup u_\infty + C \quad \text{in } L^\infty(\mathbb{R}^k) \text{ weak } * \tag{27}$$

(u_n *is supposed to be extended by* 0 *outside* Ω_n, C *denotes some constant*).

Proof. By (15), (6) we remark that $u_n - u_\infty$ satisfies

$$\begin{cases} -\partial_{x_j}\left\{a_{ij}(x)\partial_{x_i}(u_n - u_\infty)\right\} = 0 & \text{in } \Omega_n, \\ u_n - u_\infty = -u_\infty & \text{on } \partial\Omega_n. \end{cases}$$

By the maximum principle, $u_n - u_\infty$ is uniformly bounded in $\mathbb{R}^k$ and there is $v_\infty \in L^\infty(\mathbb{R}^k)$ such that – up to a subsequence:

$$u_n - u_\infty \rightharpoonup v_\infty \quad \text{in } L^\infty(\mathbb{R}^k) \text{ weak } *.$$

The above convergence takes also place in $\mathcal{D}'(\mathbb{R}^k)$ and thus it holds that

$$-\partial_{x_j}\left(a_{ij}(x)\partial_{x_i} v_\infty\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

By the Liouville theorem (see [6,7] for references) it follows that

$$v_\infty = Cst$$

which completes the proof. □

Remark 2. In the case of dimension 1 or in the case where a_{ij} are constants, one can remove the assumption (23) and show that the whole sequence u_n satisfies

$$u_n \to u_\infty + C \quad \text{in } \mathcal{D}'(\mathbb{R}^k)$$

where C can be determined, see [2].

In the case of dimension 1 and for $\epsilon_n, \delta_n \in (0, 1)$, if

$$\Omega_n = \left(-(n + \epsilon_n)T, (n + \delta_n)T\right)$$

it can happen – when ϵ_n, δ_n have no limits – that the sequence u_n has no limit.

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