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Elliptic equations with critical exponent on $\mathbf{S}^3$: new non-minimising solutions

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Abstract

Consider the problem:

$$\begin{cases} -\Delta_{S^3} U = \lambda U + U^5, \quad U > 0 & \text{on } B', \\ U = 0 & \text{on } \partial B', \end{cases}$$

where B' is a ball on $\mathbf{S}^3$ with geodesic radius θ_1, and Δ_{S^3} is the Laplace–Beltrami operator on $\mathbf{S}^3$. We prove that for any $\theta_1 \in (\pi/2, \pi)$ and any $k > 1$, there exist at least $2k$ solutions of this problem for λ sufficiently large negative. ***To cite this article: H. Brezis, L.A. Peletier, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Équations elliptiques avec exposant critique sur $\mathbf{S}^3$: nouvelles solutions non-minimisantes. On considère le problème :

$$\begin{cases} -\Delta_{S^3} U = \lambda U + U^5, \quad U > 0 & \text{sur } B', \\ U = 0 & \text{sur } \partial B', \end{cases}$$

où B' est une boule sur $\mathbf{S}^3$ de rayon geodésique θ_1, et Δ_{S^3} est l'opérateur Laplace–Beltrami sur $\mathbf{S}^3$. On montre que pour tout $\theta_1 \in (\pi/2, \pi)$, et tout $k > 1$, ce problème possède au moins $2k$ solutions pour $\lambda < 0$ avec $|\lambda|$ assez grand. ***Pour citer cet article : H. Brezis, L.A. Peletier, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

We study the Dirichlet problem on the unit sphere $\mathbf{S}^3$ in $\mathbf{R}^4$:

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$$\begin{cases} -\Delta_{S^3} U = \lambda U + U^5, \quad U > 0 & \text{on } B', \quad (1) \\ U = 0 & \text{on } \partial B'. \quad (2) \end{cases}$$

Here Δ_{S^3} is the Laplace–Beltrami operator on $\mathbf{S}^3$, and B' is the geodesic ball centered at the North pole with geodesic radius θ_1. Note that the geodesic radius of the upper half sphere is $\theta_1 = \pi/2$, and of the full sphere it is $\theta_1 = \pi$.

The exponent 5 is the Sobolev exponent in $\mathbf{R}^3$ and is known to be critical for existence of a solution.

The analogous problem in $\mathbf{R}^N$, with the Laplace–Beltrami operator replaced by the ordinary Laplacian, has been studied since 1983, when it was proposed by Brezis and Nirenberg [5]. Specifically they proved that if B' is replaced by the ball B_R with radius R, and $N = 3$, then there exists a solution if and only if

$$\frac{\pi^2}{4R^2} < \lambda < \frac{\pi^2}{R^2} = \lambda_1(-\Delta). \tag{3}$$

In recent papers by Bandle, Benguria and Peletier [3,4], it was shown that on the sphere $\mathbf{S}^3$ the situation is significantly different. They showed that in the range $\lambda > -3/4$, there is a solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2}. \tag{4}$$

Thus, in this geometry, there do exist solutions of problem (1)–(2) if $-3/4 < \lambda < 0$ and $\theta_1 > \frac{\pi}{2\sqrt{\lambda+1}}$.

For $\lambda \leqslant -3/4$ it was shown in [3], by means of a Pohozaev type identity [7], that there exist no solutions if $\theta_1 \leqslant \pi/2$, and it was conjectured in [3] that for every $\lambda < -3/4$ and every $\theta_1 < \pi$ with $\pi - \theta_1$ sufficiently small (depending on λ), a solution would indeed exist. This conjecture is still open. More recently, in [1] and [2] it was proved that given $\theta_1 \in (\pi/2, \pi)$ there exists a $\Lambda(\theta_1) < 0$ such that for every $\lambda < \Lambda(\theta_1)$ a solution exists. In addition, a detailed numerical study [8] revealed multibump solutions in the range $\lambda < -3/4$ and $\pi/2 < \theta_1 < \pi$; a family which becomes increasingly rich as $\lambda \to -\infty$. This fact is extremely interesting because in this range of λ, the minimum of the corresponding variational problem is *never* achieved.

In the present Note we only deal with *radial* solutions and denote the North pole by 0: we establish the existence of a countable family of solutions for values of λ large enough negative. Specifically we prove:

Theorem 1.1. *Given any geodesic radius $\theta_1 > \pi/2$ and any $k \geqslant 1$, then there exists a constant $A = A(k, \theta_1) > 0$ such that for $\lambda < -A$, problem* (1)–(2) *has at least $2k$ solutions, such that $U(0) \in (0, |\lambda|^{1/4})$.*

We also have strong evidence, partly numerical and partly rigorous of the following conjecture.

Conjecture. *Let*

$$\lambda_n = -\frac{1}{4}(n^2 - 1), \quad n = 2, 3, \ldots. \tag{5}$$

Let $k \geqslant 1$. Then, if $\lambda < \lambda_{2k}$, there exist at least $2k$ solutions of problem (1)–(2) *such that $U(0) < |\lambda|^{1/4}$ when the geodesic radius θ_1 of B' is sufficiently close to π.*

Remark 1. The critical numbers λ_n are, up to a factor $-1/4$, equal to the ‘radial’ eigenvalues μ_n of the eigenvalue problem

$$-\Delta_{\mathbf{S}^3} v = \mu v \quad \text{on } \mathbf{S}^3. \tag{6}$$

The radial eigenfunctions and their eigenvalues are given by

$$v_n(\theta) = \frac{\sin(n\theta)}{\sin(\theta)} \quad \text{and} \quad \mu_n = n^2 - 1, \quad n = 2, 3, \ldots. \tag{7}$$

A special role in the analysis of these solutions is played by a family of *ground states*, i.e., solutions of Eq. (1) which exist and are positive and smooth on all of $\mathbf{S}^3$. Branches of such solutions are found to emanate from the constant solution $U = |\lambda|^{1/4}$ of Eq. (1) at the special values $\lambda = \lambda_{2k+1} = -k(k+1)$, $k = 1, 2, \ldots$. These critical values correspond to the odd eigenvalues μ_{2n+1} associated with eigenfunctions which are symmetric with respect to $\theta = \pi/2$, i.e., with respect to the equatorial plane. We prove the following result about ground states:

Theorem 1.2. *Let $n \geqslant 1$, and let $\lambda < -n(n+1)$. Then there exist n ground states $U_1, \ldots, U_n$, where $U_k = u_k(\theta)$ has k local maxima ($k = 1, 2, \ldots, n$), or* spikes *on $(0, \pi)$. They are all symmetric with respect to the equatorial plane, i.e. $u_k(\theta) = u_k(\pi - \theta)$ for $0 \leqslant \theta \leqslant \pi$, and the maxima of the spikes increase with the distance from this plane.*

2. Sketch of the proofs

Using the stereographic projection $\Sigma^{-1} : \mathbf{S}^3 \to \mathbf{R}^3$ centered at the South pole, we transform the function U defined on $B' \subset \mathbf{S}^3$ to a function w on the ball $B \subset \mathbf{R}^3$: $w(x) = U(\Sigma x)$, $x \in B = B_R$. Then problem (1)–(2) becomes

$$\begin{cases} -\dfrac{1}{\rho^3}\operatorname{div}(\rho \nabla w) = \lambda w + w^5, \quad w > 0, \quad x \in B_R, & \text{(8a)} \\ w = 0, \qquad\qquad x \in \partial B_R, & \text{(8b)} \end{cases}$$

where

$$\rho(x) = \frac{2}{1 + |x|^2}, \qquad x \in B_R. \tag{9}$$

We look for solutions with radial symmetry, i.e. a solution of the form $w = w(r)$, where we see from the geodesic projection that $r = \tan(\theta/2)$ and $R = \tan(\theta_1/2)$. Thus Eq. (8a) reduces to an ordinary differential equation with the radius r as independent variable. We transform this equation once more, bringing it into the form of a generalised *Emden–Fowler* equation. Thus, we put

$$2t = \frac{1}{r} - r \quad \text{and} \quad y(t) = |\lambda|^{-1/4} w(r) \quad \text{and} \quad 2T = \frac{1}{R} - R. \tag{10}$$

Problem (8) now transforms to

$$\begin{cases} y'' + |\lambda| a(t)(y^5 - y) = 0, \quad y > 0, \quad T < t < \infty, & \text{(11a)} \\ y(T) = 0 \quad \text{and} \quad y'(\infty) = 0, & \text{(11b)} \end{cases}$$

where

$$a(t) = \frac{1}{(1 + t^2)^2}. \tag{12}$$

Note that $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$ correspond to $r = 0$, $r = 1$ and $r = \infty$, and to $t = \infty$, $t = 0$ and $t = -\infty$, and that Eq. (11a) is symmetric with respect to the origin.

The proofs of both theorems are based on a shooting argument, combined with a continuation argument, as was used in [6]. We fix $u(0) = w(0) = |\lambda|^{1/4}\gamma$ and hence $y(\infty) = \gamma$, where $0 < \gamma < 1$ is an arbitrary constant. It is well known that there then exists a unique solution $y = y(t; \gamma)$ of the problem

$$\begin{cases} y'' + |\lambda| a(t)(y^5 - y) = 0, \quad t_0 < t < \infty, & \text{(13a)} \\ y(t) \to \gamma \quad \text{as} \quad t \to \infty, & \text{(13b)} \end{cases}$$

on some interval (t_0, ∞). Since at $t = \infty$, the solution starts below the constant solution $y = 1$, its graph is convex and there will be a time $t_1 = t_1(\gamma)$, at which $y = 1$. In a left neighbourhood of t_1 the graph of y will be concave, and for $|\lambda|$ large enough it will intersect $y = 1$ again, say at the point $t_2 = t_2(\gamma)$. In the interval (t_2, t_1), the graph of y is concave, and y' has one zero, say at $\tau_1 = \tau_1(\gamma)$, a local maximum. Depending on the value of $|\lambda|$, the function $y(t) - 1$ may have further zeros $\cdots < t_4 < t_3 < t_2$, and critical points $\cdots < \tau_3 < \tau_2 < \tau_1$ in between the zeros. It is readily established that

$$t_k(\gamma) \to t_k^0 \quad \text{and} \quad \tau_k(\gamma) \to \tau_k^0 \quad \text{as } \gamma \to 1^-, \tag{14}$$

where the points t_k^0 and τ_k^0 are the zeros and the critical points of the solution of the equation we obtain by linearising (13a) about $y = 1$:

$$z'' + 4|\lambda| a(t) z = 0, \quad t \in \mathbf{R}. \tag{15}$$

This is Eq. (6) transformed to Emden–Fowler form, with μ replaced by $4|\lambda|$, so that the zeros t_k^0 and critical points τ_k^0 are all explicitly known.

As γ decreases, the critical points are all shown to move to $t = -\infty$. Hence, if a critical point starts on $\mathbf{R}^+$, i.e. if $\tau_k^0 > 0$, then it must pass the origin at some $\gamma_k \in (0, 1)$. By symmetry, we can then continue the solution $y(t; \gamma_k)$ as an even function to form a ground state on $\mathbf{R}$.

In the proof of Theorem 1.1 we put $|\lambda| = \varepsilon^{-2}$ and viewe problem (11) as a singular perturbation problem. We fix $T < 0$ and $T_0 \in (T, 0)$. Given $\varepsilon > 0$ small enough we show that there exists a $\gamma = \gamma_\varepsilon$ small enough such that $\tau_1(\gamma_\varepsilon) = T_0$. Using the energy function

$$H(t) = \frac{\varepsilon^2}{2a(t)} y'^2(t) + F\big(y(t)\big), \quad F(y) = \int_0^y (s^5 - s)\, \mathrm{d}s, \tag{16}$$

which, since

$$H'(t) = -\frac{\varepsilon^2}{2} \frac{a'(t)}{a^2(t)} y'^2(t), \tag{17}$$

is decreasing on $\mathbf{R}^-$ and increasing on $\mathbf{R}^+$, we show that $y(t; \gamma_\varepsilon)$ has a first zero $T_\varepsilon < T_0$ and that $T_\varepsilon \nearrow T_0$ as $\varepsilon \to 0$. Thus, by choosing ε small enough, we can ensure that $T_\varepsilon > T$. We now keep ε fixed and we show that $T_\varepsilon(\gamma) \to -\infty$ as $\gamma \to \gamma_\pm$, where $0 \leqslant \gamma_- < \gamma_+ < 1$, so that there will be at least two values of γ for which $T_\varepsilon(\gamma) = T$. This yields two solutions, each with one spike: one near $t = T$ and one near the origin.

Remembering the transformation (10) we can express $T_\varepsilon(\gamma)$ in terms of the radius $R_\varepsilon(\gamma)$ of the ball B_R in problem (8), and we find that $R_\varepsilon(\gamma) \to +\infty$ as $\gamma \to \gamma_\pm$.

Multi spike solutions are found is a similar manner by choosing γ_ε such that $\tau_k(\gamma_\varepsilon) = T_0$ and showing that as $\varepsilon \to 0$, the additional spikes all concentrate at the origin $t = 0$ i.e. around the equator $\theta = \pi/2$.

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