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On the distribution of the overlaps at given disorder

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Abstract

We prove a rigorous version of the following heuristic statement: if, in a spin glass model, the extended Ghirlanda–Guerra identities are valid, at given disorder the distribution of the overlap of two configurations is discrete, and its support (the smallest closed set that carries this distribution) is non-random. ***To cite this article: G. Parisi, M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur la distribution des recouvrements à désordre donné. Nous prouvons une version rigoureuse du fait suivant. Dans un modèle de verres de spins qui satisfait les identités de Ghirlanda–Guerra générales, à désordre donné, la distribution du recouvrement de deux configurations est discrète, et son support est non-aléatoire. ***Pour citer cet article : G. Parisi, M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

The study of the mean field models for spin glasses at the level of theoretical physics has required the introduction of a number of new concepts [2–4]. Despite recent progress, the rigorous (=mathematical) study of these models remains very difficult.

One of the key features of the physical picture is that at low temperatures the system governed by a mean field spin glass Hamiltonian decomposes spontaneously in series of 'pure states' (or valleys) with macroscopic Gibbs weights. It is not easy to give a mathematical definition of what this means, and even harder to conceive a program that could eventually lead to a rigorous proof of this fact. As of today, such a proof has been achieved only in cases where there is a "one level of replica-symmetry breaking" situation, such as in the p-spin interaction

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model at a suitable temperature, where the pure states are well separated from each other, and thus easier to construct [5]. Whatever the precise mathematical definition of a pure state will be, the following property will certainly be satisfied: the overlap of a generic (when weighted with the Gibbs measure) configuration in the pure state α with a generic configuration in the pure state γ should be (very close) to a number $q_{\alpha,\gamma}$ depending only on α and γ. (Precise definitions will be given below.) Consequently, if the 'valley picture' is correct, the overlap of two generic configurations is likely to be (very close to) one of a few possible values (which possibly depend on the disorder). The purpose of this Note is to prove that this remarkable property is a rigorous consequence of the extended Ghirlanda–Guerra identities (EGGI for short).

Theorem 1.1 (Heuristic version). *If an ensemble of random systems satisfies the EGGI, at given disorder, the law* (*under Gibbs' measure*) *of the overlap of* 2 *configurations is discrete, and its support is non-random.*

More precisely, we prove the following: for $\eta > 0$, all but a proportion $1 - \eta$ of this law is carried by about η^{-2} points (depending on the disorder). On the other hand, there are cases where the average over the disorder of the law of the overlaps has a continuous part. An example is rigorously constructed in [7], see also [2].

2. Precise statements

We consider a random Hamiltonian H_N on $\Sigma_N = \{-1, 1\}^N$, that depends on a parameter $\boldsymbol{\beta}$. Averages with respect to the corresponding Gibbs' measure (or its products on Σ_N^k) are denoted by $\langle\cdot\rangle$. Given configurations $\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^\ell, \ldots, \in \Sigma_N$, we define the overlaps $R_{\ell,\ell'} = N^{-1} \sum_{i \leqslant N} \sigma_i^\ell \sigma_i^{\ell'}$. We assume that the parameter $\boldsymbol{\beta}$ belongs to a compact space on which exists a probability measure $\mathrm{d}\boldsymbol{\beta}$. To simplify notation, we denote by δ a quantity depending on N and $\boldsymbol{\beta}$ such that $\lim_{N\to\infty} \int |\delta| \,\mathrm{d}\boldsymbol{\beta} = 0$. This quantity need not be the same at each occurrence.

Definition 2.1. We say that the EGGI hold if given any integer n, any continuous function $f : C_n := [-1, 1]^{n(n-1)/2} \to \mathbb{R}$ and any continuous function ϕ on $[-1, 1]$ we have, writing $g = f((R_{k,\ell})_{1 \leqslant k < \ell \leqslant n})$,

$$E\langle\phi(R_{n,n+1})g\rangle = \frac{1}{n} E\langle\phi(R_{1,2})\rangle E\langle g\rangle + \frac{1}{n} \sum_{\ell<n} E\langle\phi(R_{n+1,\ell})g\rangle + \delta. \tag{1}$$

Here and below, E denotes expectation in the randomness of the Hamiltonian H_N.

It is proved in [6], Lemma 6.4.3, that to any Hamiltonian one can add a small perturbation term, depending on a parameter $\boldsymbol{\beta}$ such that the perturbed Hamiltonian satisfies the EGGI (see also [1]). The perturbation term is small in the sense that it does not change the limiting free energy. (Unfortunately, adding this term might change the structure of the overlaps.) It is not unreasonable to think that a 'generic Hamiltonian' satisfies the EGGI without the addition of a perturbation term (or equivalently, when $\boldsymbol{\beta}$ takes only one value) but often specific Hamiltonians do not.

To express that a probability measure is almost supported by a few points we introduce the following definition.

Definition 2.2. For a probability measure μ on $[-1, 1]$, an integer n and $\epsilon > 0$ we define $A(\mu, n, \epsilon)$ as the maximum amount of mass of μ that can be carried by the union of n intervals of length at most 2ϵ, i.e.

$$A(\mu, n, \epsilon) = \sup\bigl\{\mu(B);\ B \subset [-1, 1] \text{ is the union of } n \text{ intervals of length} \leqslant 2\epsilon\bigr\}.$$

We can now give the precise formulation of Theorem 1.1.

Theorem 2.3. *If a random Hamiltonian satisfies the EGGI and μ denotes the* (*random*) *law of $R_{1,2}$ under Gibbs' measure, for each integer n and each $\epsilon > 0$ we have*

$$E\left(A\left(\mu, \frac{n(n-1)}{2}, \epsilon\right)\right) \geqslant 1 - \frac{2}{n+1} + \delta. \tag{2}$$

Moreover, if ϕ is a continuous function with $0 \leqslant \phi \leqslant 1$, for each n we have

$$P\left(\int \phi \,\mathrm{d}\mu \geqslant \frac{1}{n}\right) \geqslant 1 - \frac{8}{(\log n) E \int \phi \,\mathrm{d}\mu} + \delta. \tag{3}$$

Of course, we have $\int \phi \,\mathrm{d}\mu = \langle \phi(R_{1,2}) \rangle$, and E denotes the expectation for the disorder.

In other words, if N is large, for typical $\boldsymbol{\beta}$ and typical disorder, n^2 very small intervals suffice to carry all but about $1/n$ of the mass of μ. If the open interval I meets the support of the average of μ over the disorder, then (if N is large, for typical $\boldsymbol{\beta}$ and typical disorder), we have $\mu(I) > 0$.

3. Proofs

Lemma 3.1. *If the EGGI hold, given any continuous function $f : C_n \to \mathbb{R}$ and any continuous function ϕ on $[-1, 1]$ we have, using the notation $g = f((R_{k,\ell})_{1 \leqslant k < \ell \leqslant n})$,*

$$E\langle \phi(R_{n+1,n+2}) g \rangle = \frac{2}{n+1} E\langle \phi(R_{1,2}) \rangle E\langle g \rangle + \frac{1}{n(n+1)} \sum_{k \neq \ell,\ k,\ell \leqslant n} E\langle \phi(R_{k,\ell}) g \rangle + \delta. \tag{4}$$

Proof. We use (1) with $n + 1$ rather than n to get

$$E\langle \phi(R_{n+1,n+2}) g \rangle = \frac{1}{n+1} E\langle \phi(R_{1,2}) \rangle E\langle g \rangle + \frac{1}{n+1} \sum_{\ell \leqslant n} E\langle \phi(R_{n+1,\ell}) g \rangle + \delta. \tag{5}$$

We use (1) again to get that for each $\ell \leqslant n$ we have

$$E\langle \phi(R_{n+1,\ell}) g \rangle = \frac{1}{n} E\langle \phi(R_{1,2}) \rangle E\langle g \rangle + \frac{1}{n} \sum_{k \neq \ell,\ k,\ell \leqslant n} E\langle \phi(R_{k,\ell}) g \rangle + \delta,$$

and we substitute in (5). □

Let us denote by μ the law of $R_{1,2}$ in $[-1, 1]$ under Gibbs' measure and by ν the law of $(R_{k,\ell})_{1 \leqslant k < \ell \leqslant n}$ in C_n under Gibbs' measure. Let us further denote by $\overline{\mu}$ and $\bar{\nu}$ the averages of μ and ν respectively with respect to the disorder. We denote by $\boldsymbol{x} = (x_{k,\ell})_{1 \leqslant k < \ell < n}$ the generic point of C_n. Then (4) means that

$$\begin{aligned} E \int \phi(x) f(\boldsymbol{x}) \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(\boldsymbol{x}) &= \frac{2}{n+1} \int \phi(x) f(\boldsymbol{x}) \,\mathrm{d}\bar{\mu}(x) \,\mathrm{d}\bar{\nu}(\boldsymbol{x}) \\ &\quad + \frac{1}{n(n+1)} \sum_{k \neq \ell,\ k,\ell \leqslant n} \int \phi(x_{k,\ell}) f(\boldsymbol{x}) \,\mathrm{d}\bar{\mu}(\boldsymbol{x}) + \delta. \end{aligned} \tag{6}$$

Proof of Theorem 2.3. Consider the function ψ on $[-1, 1] \times C_n$ given by

$$\psi\big((x, \boldsymbol{x})\big) = \min\left(1, \frac{1}{\epsilon} \min_{1 \leqslant k < \ell \leqslant n} |x - x_{k,\ell}|\right). \tag{7}$$

Since this function is continuous it can be approximated arbitrarily well by a finite sum of functions of the type $\phi(x) f(\boldsymbol{x})$ where ϕ and f are continuous, so that by (6) we have

$$\begin{aligned} E \int \psi(x, \boldsymbol{x}) \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(\boldsymbol{x}) &= \frac{2}{n+1} \int \psi(x, \boldsymbol{x}) \,\mathrm{d}\bar{\mu}(x) \,\mathrm{d}\bar{\nu}(\boldsymbol{x}) \\ &\quad + \frac{1}{n(n+1)} \sum_{k \leqslant \ell,\ k,\ell \leqslant n} \int \psi(x_{k,\ell}, \boldsymbol{x}) \,\mathrm{d}\bar{\mu}(\boldsymbol{x}) + \delta. \end{aligned} \tag{8}$$

Since $\psi(x_{k,\ell}, \boldsymbol{x}) = 0$ we get that $E \int \psi(x, \boldsymbol{x}) \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(\boldsymbol{x}) \leqslant \frac{2}{n+1} + \delta$, and in particular that

$$E \inf_{\boldsymbol{x}} \int \psi(x, \boldsymbol{x})\, \mathrm{d}\mu(x) \leqslant \frac{2}{n+1} + \delta.$$

Since $0 \leqslant \psi \leqslant 1$, for each $\boldsymbol{x}$ we have

$$1 - \mu\big(\{x;\, \psi(x, \boldsymbol{x}) < 1\}\big) = \mu\big(\{x;\, \psi(x, \boldsymbol{x}) = 1\}\big) \leqslant \int \psi(x, \boldsymbol{x})\, \mathrm{d}\mu(x),$$

so that

$$1 - \frac{2}{n+1} - \delta \leqslant E \sup_{\boldsymbol{x}} \mu\big(\{x;\, \psi(x, \boldsymbol{x}) < 1\}\big).$$

Since

$$\{x;\, \psi(x, \boldsymbol{x}) < 1\} \subset \bigcup_{1 \leqslant k < \ell \leqslant n} [x_{k,\ell} - \epsilon, x_{k,\ell} + \epsilon],$$

for each $\boldsymbol{x}$ we have

$$\mu\big(\{x, \psi(x, \boldsymbol{x}) < 1\}\big) \leqslant A\left(\mu, \frac{n(n+1)}{2}, \epsilon\right)$$

and this concludes the proof of (2). To prove (3), we use (4) for $n = 2m$, taking $f(\boldsymbol{x}) = \prod_{1 \leqslant k \leqslant m} (1 - \phi(x_{2k-1,2k}))$, to get

$$E\left(\int \phi\, \mathrm{d}\mu \left(1 - \int \phi\, \mathrm{d}\mu\right)^m\right) \geqslant \frac{2}{2m+1} E\left(\int \phi\, \mathrm{d}\mu\right) E\left(1 - \int \phi\, \mathrm{d}\mu\right)^m + \delta.$$

Since $a \sum_{m \geqslant 1} (1-a)^m \leqslant 1$, summation for $m \leqslant p$ yields

$$(\log p) E\left(\int \phi\, \mathrm{d}\mu\right) E\left(1 - \int \phi\, \mathrm{d}\mu\right)^p \leqslant 2 + \delta,$$

so that, since $(1 - 1/p)^p \geqslant 1/4$,

$$\frac{1}{4} (\log p) E\left(\int \phi\, \mathrm{d}\mu\right) P\left(\int \phi\, \mathrm{d}\mu \leqslant \frac{1}{p}\right) \leqslant 2 + \delta. \qquad \square$$

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