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Differential Geometry

On the recovery and continuity of a submanifold with boundary in higher dimensions

Marcela Szopos

Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France

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Abstract

Let ω be a connected and simply connected open subset of $\mathbb{R}^p$ endowed with a Riemannian metric. Under a smoothness assumption on the boundary of ω, we first establish the existence and uniqueness up to isometries of an isometric immersion of ω into the Euclidean space $\mathbb{R}^{p+q}$, 'up to the boundary' of ω. When ω is bounded, we also show that the mapping that associates with the prescribed geometrical data the reconstructed submanifold is locally Lipschitz-continuous with respect to the topology of the Banach spaces $\mathcal{C}^l(\overline{\omega}),\ l \geqslant 1$. ***To cite this article: M. Szopos, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur la reconstruction et la continuité d'une sous-variété à bord en dimensions supérieures. Soit ω un ouvert connexe et simplement connexe de $\mathbb{R}^p$, muni d'une métrique riemannienne. Sous une certaine hypothèse de régularité sur la frontière de ω, on établit d'abord l'existence et l'unicité aux isométries près d'une immersion isométrique de ω dans l'espace euclidien $\mathbb{R}^{p+q}$, « jusqu'au bord » de ω. Lorsque ω est borné, on montre aussi que l'application qui associe aux données géométriques prescrites la sous-variété ainsi reconstruite est localement lipschitzienne pour les topologies usuelles des espaces de Banach $\mathcal{C}^l(\overline{\omega}),\ l \geqslant 1$.
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Version française abrégée

Soit $\omega \subset \mathbb{R}^p$ un ouvert connexe et simplement connexe, muni d'une métrique riemannienne. Il est connu en géométrie différentielle (voir Théorème 2.1), que ω peut être isométriquement immergé dans l'espace euclidien

E-mail address: szopos@ann.jussieu.fr (M. Szopos).

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identifié à $\mathbb{R}^{p+q}$ si et seulement si les tenseurs associés vérifient les relations de Gauss–Ricci–Codazzi (3)–(5) et que les immersions ainsi définies sont déterminées aux isométries de $\mathbb{R}^{p+q}$ près.

Dans des travaux récents (voir [3] pour le cas d'un ouvert de $\mathbb{R}^n$ et [4] pour le cas d'une surface), on étudie dans quelles conditions ce type de résultat reste vrai « jusqu'au bord » de l'ouvert de départ. Notre objectif est de généraliser les résultats mentionnés précédemment au cas des immersions de dimension et de co-dimension quelconques.

Cette étude est motivée par des questions rencontrées dans la théorie de l'élasticité. De ce point de vue, l'immersion $\theta:\omega\to\mathbb{R}^{p+q}$ peut être interprétée comme la déformation d'un solide qui occupe l'ensemble ω sous l'effet des forces extérieures (à condition que l'application θ soit, de plus, injective, car on veut éviter l'interpénétrabilité de la matière). L'intérêt d'étudier les aspects géométriques de la mécanique des solides en dimension quelconque est souligné, par exemple, en [1].

Le premier objectif de cette Note est d'établir, sous une certaine hypothèse de régularité sur la frontière de ω (voir Définition 1.1 pour la propriété géodésique), un résultat d'existence et unicité d'une sous-variété « jusqu'au bord » (voir Théorème 2.2). Comme conséquence de ce résultat, on peut définir une application qui associe aux données géométriques de départ la sous-variété ainsi construite.

Le deuxième objectif de cette Note est de montrer que, lorsque l'ouvert ω est borné, cette application est localement lipschitzienne pour les topologies usuelles des espaces de Banach $\mathcal{C}^l(\overline{\omega})$, $l\geqslant 1$ (au sens de la Définition 1.2).

Les démonstrations complètes des résultats de cette Note se trouvent dans [6].

1. Preliminaries

Let an integer $N\geqslant 1$ be fixed. The notations $\mathbb{M}^N$, $\mathbb{O}^N$, $\mathbb{S}^N$, $\mathbb{A}^N$ and $\mathbb{S}^N_>$, respectively, designate the set of all real square matrices, of all orthogonal matrices, of all symmetric matrices, of all anti-symmetric matrices and of all positive definite symmetric matrices of order N. We denote by $|A|$ the operator norm of a matrix $A\in\mathbb{M}^N$ and by I_N the identity matrix of order N.

The results of this Note are established under a specific, but mild, regularity assumption on the boundary $\partial\omega$ of an open subset ω of $\mathbb{R}^p$, $p\geqslant 1$, according to the following definition:

Definition 1.1. An open subset ω of $\mathbb{R}^p$ satisfies the geodesic property if it is connected and, given any point $x_0\in\partial\omega$ and any $\epsilon>0$, there exists $\delta=\delta(x_0,\epsilon)>0$ such that

$$d_\omega(x,y)<\epsilon \quad \text{for all } x,y\in\omega\cap B(x_0;\delta),$$

where $B(x_0;\delta)$ denotes the open ball with center x_0 and radius $\delta>0$, and $d_\omega(x,y):=\inf\{\text{length}(\gamma),\ \gamma \text{ is a path joining } x \text{ to } y \text{ in } \omega\}$.

Remark 1. Notice that this assumption is not a very restrictive one, since any connected open subset of $\mathbb{R}^p$ with a Lipschitz-continuous boundary satisfies the geodesic property.

We introduce in the following definition the notion of spaces of functions, vector fields, or matrix fields 'of class $\mathcal{C}^l$ up to the boundary of ω'.

Definition 1.2. Let ω be an open subset of $\mathbb{R}^p$. For any integer $l\geqslant 1$, we define the space $\mathcal{C}^l(\overline{\omega})$ as the space of all functions $f\in\mathcal{C}^l(\omega)$ that, together with all their partial derivatives $\partial^\alpha f$, $1\leqslant|\alpha|\leqslant l$, possess continuous extensions to the closure $\overline{\omega}$ of ω. Analogous definitions hold for the spaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{S}^N)$, etc., for all integer $N>0$. All the continuous extensions appearing in such spaces will be identified by a bar, as exemplified in the definition of the following set:

$$\mathcal{C}^2\big(\overline{\omega};\mathbb{S}^N_>\big):=\big\{A\in\mathcal{C}^l\big(\overline{\omega};\mathbb{S}^N\big);\ \bar{A}(y)\in\mathbb{S}^N_> \text{ for all } y\in\overline{\omega}\big\}.$$

The spaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$ and $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $l\geqslant 1$, are endowed with their natural norms, defined by:

$$\|F\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\alpha|\leqslant l} \left|\overline{\partial^\alpha F}(y)\right| \quad \text{for all } F\in\mathcal{C}^l\big(\overline{\omega};\mathbb{M}^N\big),$$

and

$$\|\theta\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\alpha|\leqslant l} \left|\overline{\partial^\alpha \theta}(y)\right| \quad \text{for all } \theta\in\mathcal{C}^l\big(\overline{\omega};\mathbb{R}^N\big).$$

2. Recovery of a submanifold with boundary

Let two integers $p\geqslant 1$, $q\geqslant 0$ be fixed. Let ω be an open subset of $\mathbb{R}^p$. We denote by $y=(y_1,\dots,y_p)\in\omega$ a generic point of ω and by $\{e_1,\dots,e_p\}$ the canonical basis of $\mathbb{R}^p$. In what follows, Greek indices vary in the set $\{1,\dots,p\}$, Latin indices vary in the set $\{1,\dots,q\}$ and the summation convention with respect to repeated indices is systematically used. The inner product in $\mathbb{R}^{p+q}$ is denoted $\langle\cdot,\cdot\rangle$.

Let $\theta:\omega\to\mathbb{R}^{p+q}$ be an immersion of the open set ω in the $(p+q)$-dimensional Euclidean space, identified with $\mathbb{R}^{p+q}$. Then, for each $y\in\omega$, there exists a neighborhood $U\subset\omega$ of y such that $\theta(U)\subset\mathbb{R}^{p+q}$ is a submanifold of $\mathbb{R}^{p+q}$ (for details, see [2, Chapter 6, Section 2]). A basis in the tangent space $T_{\theta(y)}\theta(\omega)$ is given by

$$\partial_1\theta(y) := \mathrm{d}\theta_y(e_1),\dots,\partial_p\theta(y) := \mathrm{d}\theta_y(e_p),$$

a vector field N is called *normal* if $\langle N(y),\partial_\alpha\theta(y)\rangle=0$ for all $y\in\omega$ and for all $\alpha\in\{1,\dots,p\}$. The derivatives of N are given by $\partial_\alpha N(y) := \mathrm{d}N_y(e_\alpha)$.

We denote by $\mathcal{X}(\omega)$ and $\mathcal{X}(\omega)^\perp$ the set of differentiable vector fields that are tangent, respectively normal, to $\theta(\omega)$.

The Euclidean metric of $\mathbb{R}^{p+q}$ induces a Riemannian metric on ω, also called the *first fundamental form* of the immersion, defined by its covariant components

$$a_{\alpha\beta}(y) := \big\langle\partial_\alpha\theta(y),\partial_\beta\theta(y)\big\rangle,\quad \forall y\in\omega,\ \forall\alpha,\beta\in\{1,\dots,p\}.$$

If ω is endowed with this metric, θ becomes an isometric immersion of ω into $\mathbb{R}^{p+q}$.

Let $\overline{\nabla}$ denote the usual Riemann connection on $\mathbb{R}^{p+q}$ (see [2, Chapter 2, Section 3]). If X and Y are vector fields on ω and $\overline{X}$, $\overline{Y}$ are local extensions to $\mathbb{R}^{p+q}$, we define the *induced Riemann connection* on $\theta(\omega)$ by $\nabla_X Y := (\overline{\nabla}_{\overline{X}}\overline{Y})^T$, where by V^T we denote the tangential part of a vector V. The remaining normal part is denoted $B(X,Y)$. The mapping $B:\mathcal{X}(\omega)\times\mathcal{X}(\omega)\to\mathcal{X}^\perp(\omega)$ defined in this fashion is a symmetric bilinear mapping, which is called the *second fundamental form* of the immersion $\theta:\omega\to\mathbb{R}^{p+q}$. We thus have:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{1}$$

for all tangent vector fields X and Y.

To define the induced *normal connection* of the immersion, we take the projection of $\overline{\nabla}_X N$ onto the normal space, where X is a tangent vector field and N is a normal vector field. Denoting this induced connection by $\nabla_X^\perp N$, we thus have:

$$\overline{\nabla}_X N = A(X,N) + \nabla_X^\perp N, \tag{2}$$

where the mapping $A:\mathcal{X}(\omega)\times\mathcal{X}^\perp(\omega)\to\mathcal{X}(\omega)$ is related to the mapping B by the relation $\langle A(X,N),Y\rangle = -\langle B(X,Y),N\rangle$ for all $Y\in\mathcal{X}(\omega)$.

We now write these expressions in the local coordinates $(y_1,\dots,y_p)$ on ω. Denote by $X_\alpha=\partial_\alpha\theta$, for all $1\leqslant\alpha\leqslant p$, the basis in the tangent bundle and fix an orthogonal basis, denoted $\{N^1,\dots,N^q\}$, in the normal bundle. Hence: $\langle N^i,N^j\rangle=\delta^{ij}$ and $\langle N^i,X_\alpha\rangle=0$ for all $1\leqslant i,j\leqslant q$ and $1\leqslant\alpha\leqslant p$.

The Riemann connection induced on $\theta(\omega)$ and the normal connection of the immersion define the *Christoffel symbols* by the following relations: $\nabla_{X_\alpha}X_\beta = \Gamma^\tau_{\alpha\beta}X_\tau$ and $\nabla^\perp_{X_\beta}N^i = T^{ij}_\beta N^j$.

The coefficients of the second fundamental form are defined by $b^i_{\alpha\beta} := -\langle B(X_\alpha, X_\beta), N^i\rangle$. Note that the mapping A can be expressed in local coordinates as follows: $A(X_\alpha, N^i) = b^{i\tau}_\alpha X_\tau$, where $b^{i\tau}_\alpha := a^{\sigma\tau} b^i_{\alpha\sigma}$ denote the mixed components of the second fundamental form and $(a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$.

Since the Euclidean space $\mathbb{R}^{p+q}$ is flat, the Riemann curvature tensor associated with the connection $\overline{\overline{\nabla}}$ on $\mathbb{R}^{p+q}$ vanishes. Using relations (1) and (2) and expressing the fact that its tangential part and its normal part vanish, we deduce the following relations, written in local coordinates:

$$\big(\partial_\alpha \Gamma^\tau_{\beta\delta} - \partial_\beta \Gamma^\tau_{\alpha\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\tau_{\alpha\sigma} - \Gamma^\sigma_{\alpha\delta}\Gamma^\tau_{\beta\sigma}\big) a_{\tau\gamma} = b^i_{\gamma\alpha} b^i_{\delta\beta} - b^i_{\gamma\beta} b^i_{\delta\alpha}, \tag{3}$$

$$\partial_\alpha T^{ij}_\beta - \partial_\beta T^{ij}_\alpha + T^{kj}_\beta T^{ik}_\alpha - T^{kj}_\alpha T^{ik}_\beta + a^{\sigma\tau}\big(b^j_{\alpha\tau} b^i_{\beta\sigma} - b^j_{\beta\tau} b^i_{\alpha\sigma}\big) = 0, \tag{4}$$

$$\partial_\alpha b^j_{\gamma\beta} - \partial_\beta b^j_{\gamma\alpha} = \Gamma^\tau_{\alpha\gamma} b^j_{\tau\beta} - \Gamma^\tau_{\beta\gamma} b^j_{\tau\alpha} + b^i_{\gamma\beta} T^{ij}_\alpha - b^i_{\gamma\alpha} T^{ij}_\beta, \tag{5}$$

where $\alpha, \beta, \gamma, \delta, \sigma, \tau \in \{1, \ldots, p\}$ and $i, j, k \in \{1, \ldots, q\}$. These are the classical Gauss–Ricci–Codazzi equations satisfied for a submanifold of Euclidean space, as given, for example, in [5].

The following theorem establishes the existence, and uniqueness under some additional assumptions, of an isometric immersion of an open subset $\omega \subset \mathbb{R}^p$ endowed with a Riemannian metric into the Euclidean space $\mathbb{R}^{p+q}$. It is a classical result, save for the choice of conditions at y_0: a proof using local coordinates can be found, for example, in [5], which also includes a general discussion on this subject.

Theorem 2.1. *Let ω be a connected and simply connected open subset of $\mathbb{R}^p$. Let there be given a positive definite symmetric matrix field $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^p_>)$, q symmetric matrix fields $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^p)$, $i \in \{1, \ldots, q\}$, and p anti-symmetric matrix fields $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\omega; \mathbb{A}^q)$, $\alpha \in \{1, \ldots, p\}$, satisfying the Gauss–Ricci–Codazzi equations* (3)–(5). *Let $y_0 \in \omega$ be fixed and denote by $G_0 \in \mathbb{M}^{p+q}$ the following matrix*:

$$G_0 := \begin{pmatrix} A(y_0) & 0 \\ 0 & I_q \end{pmatrix}. \tag{6}$$

Then there exist a unique immersion $\theta \in \mathcal{C}^3(\omega, \mathbb{R}^{p+q})$ and a unique orthonormal family of q vector fields $N^1, \ldots, N^q \in \mathcal{C}^2(\omega, \mathbb{R}^{p+q})$, normal to $\theta(\omega)$, satisfying

$$\text{(i)}\quad \big\langle \partial_\alpha\theta(y), \partial_\beta\theta(y)\big\rangle = a_{\alpha\beta}(y) \quad \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \ldots, p\}, \tag{7}$$

$$\text{(ii)}\quad \big\langle \partial_{\alpha\beta}\theta(y), N^i(y)\big\rangle = -b^i_{\alpha\beta}(y) \quad \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \ldots, p\},\ \forall i \in \{1, \ldots, q\}, \tag{8}$$

$$\text{(iii)}\quad \big\langle \partial_\alpha N^i(y), N^j(y)\big\rangle = T^{ij}_\alpha(y) \quad \forall y \in \omega,\ \forall \alpha \in \{1, \ldots, p\},\ \forall i, j \in \{1, \ldots, q\}, \tag{9}$$

such that $\theta(y_0) = 0$ and $\partial_\alpha\theta(y_0)$ is the α-th column of the matrix $G_0^{1/2}$ for all $\alpha \in \{1, \ldots, p\}$.

The first objective of this Note is to establish that, under *ad hoc* assumptions, the reconstruction of a submanifold of $\mathbb{R}^{p+q}$ can be done ‘up to the boundary’. More specifically, the existence and uniqueness of the isometric immersion $\theta : \omega \subset \mathbb{R}^p \to \mathbb{R}^{p+q}$ recalled in Theorem 2.1 can be extended as follows to the closure $\overline{\omega}$ of the subset $\omega \subset \mathbb{R}^p$. This theorem generalizes previous results obtained in [3] and [4].

Theorem 2.2. *Let ω be a simply connected open subset of $\mathbb{R}^p$ that satisfies the geodesic property* (*Definition* 1.1). *Consider a positive definite symmetric matrix field $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$, q symmetric matrix fields $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $i \in \{1, \ldots, q\}$, and p anti-symmetric matrix fields $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\alpha \in \{1, \ldots, p\}$, such that relations* (3)–(5) *are satisfied. Then there exist a mapping $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1, \ldots, N^q \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$, normal to $\theta(\overline{\omega})$, satisfying*

$$\text{(i)}\quad \big\langle \overline{\partial_\alpha\theta}(y), \overline{\partial_\beta\theta}(y)\big\rangle = \overline{a_{\alpha\beta}}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \ldots, p\}, \tag{10}$$

(ii) $\langle \overline{\partial_{\alpha\beta}\theta}(y), \overline{N^i}(y)\rangle = -\overline{b^i_{\alpha\beta}}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\},$ (11)

(iii) $\langle \overline{\partial_\alpha N^i}(y), \overline{N^j}(y)\rangle = \overline{T^{ij}_\alpha}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i, j \in \{1, \dots, q\},$ (12)

such that $\theta(y_0) = 0$ *and* $\partial_\alpha \theta(y_0)$ *is the* α*-th column of the matrix* $G_0^{1/2}$ *for all* $\alpha \in \{1, \dots, p\}$*, where the matrix* G_0 *is given by relation* (6).

Sketch of proof. Theorem 2.1 shows that there exist a mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1, \dots, N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$, normal to $\theta(\omega)$, satisfying the required conditions in ω. In order to establish Theorem 2.2, the idea is to prove that these applications can be extended by continuity to $\overline{\omega}$. To this end, we use the fact that ω satisfies the geodesic property to derive some estimates of Gronwall's type that eventually lead to the desired extensions. □

3. Continuity of a submanifold with boundary

Let ω be a simply-connected open subset of $\mathbb{R}^p$ that satisfies the geodesic property. We introduce the following set of matrix fields:

$$X(\overline{\omega}) := \big\{ \big(A, \big(B^i\big), (T_\alpha)\big) \in \mathcal{C}^2\big(\overline{\omega}; \mathbb{S}^p_>\big) \times \big(\mathcal{C}^1\big(\overline{\omega}; \mathbb{S}^p\big)\big)^q \times \big(\mathcal{C}^1\big(\overline{\omega}; \mathbb{A}^q\big)\big)^p \\ \text{such that relations (3)–(5) are satisfied}\big\}.$$

Fix a point $y_0 \in \omega$ and consider the matrix G_0 given by relation (6). Then by Theorem 2.2, there exists a well-defined mapping

$$\mathcal{F} : X(\overline{\omega}) \to \mathcal{C}^3\big(\overline{\omega}; \mathbb{R}^{p+q}\big) \times \big(\mathcal{C}^2\big(\overline{\omega}; \mathbb{R}^{p+q}\big)\big)^q$$

that associates with any point $(A, (B^i), (T_\alpha)) \in X(\overline{\omega})$ the uniquely determined element $(\theta, (N^i)) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times (\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q}))^q$ satisfying

$$\langle \overline{\partial_\alpha\theta}(y), \overline{N^i}(y)\rangle = 0 \quad \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\},$$
$$\langle \overline{\partial_\alpha\theta}(y), \overline{\partial_\beta\theta}(y)\rangle = \overline{a_{\alpha\beta}}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \dots, p\},$$
$$\langle \overline{\partial_{\alpha\beta}\theta}(y), \overline{N^i}(y)\rangle = -\overline{b^i_{\alpha\beta}}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\},$$
$$\langle \overline{\partial_\alpha N^i}(y), \overline{N^j}(y)\rangle = \overline{T^{ij}_\alpha}(y) \quad \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i, j \in \{1, \dots, q\},$$

such that $\theta(y_0) = 0$ and $\partial_\alpha\theta(y_0)$ is the α-th column of the matrix $G_0^{1/2}$ for all $\alpha \in \{1, \dots, p\}$. The second objective of this Note is to study the continuity of this mapping.

We first note that, if in addition the set ω is bounded, the spaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ and $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$, endowed with their natural norms, become Banach spaces. The product space $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times (\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p))^q \times (\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q))^p$, endowed with the norm

$$\big\|\big(A, \big(B^i\big), (T_\alpha)\big)\big\|_{2,1,1,\overline{\omega}} = \|A\|_{2,\overline{\omega}} + \max_i \big\|B^i\big\|_{1,\overline{\omega}} + \max_\alpha \|T_\alpha\|_{1,\overline{\omega}}$$

becomes a Banach space and thus in this case the set $X(\overline{\omega})$ becomes a metric space when it is equipped with the induced topology. The product space $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times (\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q}))^q$, endowed with the norm

$$\big\|\big(\theta, \big(N^i\big)\big)\big\|_{3,2,\overline{\omega}} = \|\theta\|_{3,\overline{\omega}} + \max_i \big\|N^i\big\|_{2,\overline{\omega}}$$

is also a Banach space. We are now in a position to state the continuity result, which generalizes the results obtained in [3] and [4].

Theorem 3.1. *Let ω be a simply-connected and bounded open subset of $\mathbb{R}^p$ that satisfies the geodesic property. Let the spaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times (\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p))^q \times (\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q))^p$ and $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times (\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q}))^q$ be endowed with the norms $\|\cdot\|_{2,1,1,\overline{\omega}}$ and $\|\cdot\|_{3,2,\overline{\omega}}$ respectively, and let the set $X(\overline{\omega})$ be equipped with the metric induced by the norm $\|\cdot\|_{2,1,1,\overline{\omega}}$. Then the mapping*

$$\mathcal{F}: \big(A, \big(B^i\big), (T_\alpha)\big) \in X(\overline{\omega}) \to \big(\theta, \big(N^i\big)\big) \in \mathcal{C}^3\big(\overline{\omega}; \mathbb{R}^{p+q}\big) \times \big(\mathcal{C}^2\big(\overline{\omega}; \mathbb{R}^{p+q}\big)\big)^q$$

is continuous. It is even locally Lipschitz-continuous over the set $X(\overline{\omega})$.

Sketch of proof. A generic element of the space $X(\overline{\omega})$ is denoted $X := (A, (B^i), (T_\alpha))$. Let $\widehat{X} := (\hat{A}, (\widehat{B}^i), (\widehat{T}_\alpha))$ be an arbitrary point of the space $X(\overline{\omega})$. The aim is to prove that there exist constants $c(\widehat{X}) > 0$ and $\delta(\widehat{X}) > 0$ such that

$$\big\|\big(\theta, \big(N^i\big)\big) - \big(\tilde{\theta}, \big(\widetilde{N}^i\big)\big)\big\|_{3,2,\overline{\omega}} \leqslant c(\widehat{X})\|X - \widetilde{X}\|_{2,1,1,\overline{\omega}}$$

for all $X, \widetilde{X} \in B(\widehat{X}; \delta(\widehat{X}))$, where $(\theta, (N^i)) = \mathcal{F}(X)$ and $(\tilde{\theta}, (\widetilde{N}^i)) = \mathcal{F}(\widetilde{X})$. The strategy then consists in writing $\mathcal{F}$ as a composite mapping and showing that each one of the factor mappings is locally Lipschitz-continuous. To this end, we use the continuous dependence on the data of appropriate Cauchy problems, associated with the mappings $(\theta, (N^i))$ and $(\tilde{\theta}, (\widetilde{N}^i))$. □

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