Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 215–218
COMPTES RENDUS MATHEMATIQUE

Differential Topology

Lefschetz pencil structures for 2-calibrated manifolds

Alberto Ibort, David Martínez Torres

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés, Spain

Received 29 March 2004; accepted after revision 20 May 2004

Presented by Étienne Ghys

Abstract

We prove that for closed 2-calibrated manifolds there always exist Lefschetz pencil structures. This generalizes similar results for symplectic and contact manifolds. ***To cite this article: A. Ibort, D. Martínez Torres, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Pinceaux de Lefschetz pour variétés 2-calibrées. On prouve qu'il existe toujours des pinceaux de Lefschetz pour les variétés fermées 2-calibrées. Ce résultat généralise des constructions similaires pour les variétés symplectiques et de contact. ***Pour citer cet article : A. Ibort, D. Martínez Torres, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

The programme started by Donaldson [3] by using approximately holomorphic geometry to study the topology of symplectic manifolds has been extended recently to the wider framework of 2-calibrated manifolds (see [7] and references therein). In the same vein Donaldson was able to show the existence of Lefschetz pencils structures for closed symplectic manifolds, extending in this way well-known ideas in Hodge geometry to the symplectic category. In this Note we will show the existence of the corresponding notion of Lefschetz pencils for 2-calibrated structures generalizing the construction by Presas [9] for contact manifolds (see also the related open book decompositions introduced in [6]).

A 2-calibrated manifold is a smooth manifold M of dimension $2n+1$ carrying a codimension 1 distribution D, and a closed 2-form ω non-degenerate on D [7]. The notion of 2-calibrated manifold includes as particular instances those of contact manifolds, certain constant rank Poisson manifolds, positive confoliations, and as a very particular instance smooth taut foliations of 3-manifolds. Thinking precisely on the later example, we could imagine a Lefschetz pencil as a map defining on each leaf a ramified covering of the Riemann sphere by a nearly holomorphic

E-mail addresses: albertoi@math.uc3m.es (A. Ibort), dmtorres@math.uc3m.es (D. Martínez Torres).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.05.018

map with singular points of index 2. Thus the ramification set would be given by a link transversal to the foliation and such that its image would be in general position. A natural generalization for 2-calibrated manifolds is (see also [9]):

Definition 1.1. Let (M, D, ω) be a 2-calibrated manifold. A Lefschetz pencil structure on M is given by a triple (f, A, B), where B is a compact real codimension four 2-calibrated submanifold, and $f : M\backslash B \to S^2$ is a smooth map whose set of singular points along D is the compact 2-calibrated one-dimensional submanifold A. The image of A (with possibly many connected components) has to be an immersed curve with transverse double points. Moreover, f has the following local structure:

(i) For each point $x \in B$ there are coordinates $z_1, \dots, z_n, s$ compatible with (ω, D) centered at x and a holomorphic chart in $\mathbb{CP}^1$, such that in a ball small enough the set B is given by $z_1 = z_2 = 0$, and out of B the map is given by $f(z, s) = z_2/z_1$.
(ii) For each point $y \in A$ there are coordinates $z_1, \dots, z_n, s$ compatible with (ω, D) centered at y and a holomorphic chart in $\mathbb{CP}^1$, such that $f(z, s) = g(s) + z_1^2 + \cdots + z_n^2$, where $g(0) = f(y)$ and $g'(0) \neq 0$.

The inverse image of a regular point c is thus an open 2-calibrated submanifold in $M\backslash A$. Because of the local model around points in B, the closure of $f^{-1}(c)$ is the closed calibrated submanifold $f^{-1}(c) \cup B$, that will be called the fibre of f at c. We can state now the main result of this note.

Theorem 1.2. *Any closed* 2*-calibrated manifold has a Lefschetz pencil structure.*

Applying a slightly refined version of this result to a smooth taut foliation $\mathcal{F}$ of M^3 we get:

Corollary 1.3. *There exist a pair* (f, A) *where* A *is a link transversal to* $\mathcal{F}$ *and* $f : M^3 \to S^2$ *is a smooth map which* (i) *is a submersion along* $T\mathcal{F}$ *on* $M\backslash A$, (ii) $f(A)$ *is in general position and* (iii) *around any point* $x \in A$ *there exist coordinates* z, s *compatible with* $(\omega, \mathcal{F})$, *and a complex coordinate in* S^2 *such that* $f(z, s) = g(s) + z^2$, *with* $g(0) = 0$, $g'(0) \neq 0$.

In dimension 3 the foliation is by Riemann surfaces. Our techniques do not produce in general meromorphic functions. These have been constructed by Ghys [5] for laminations by Riemann surfaces of compact spaces (see also [2] for more general results), but they are not generic in our sense. It will be shown elsewhere that when $(M, D, \omega) \subset \mathbb{CP}^N$, D is a distribution by complex subspaces and ω is the restriction of the Fubini–Study 2-form, an approximately holomorphic theory yielding holomorphic results can be developed.

2. Local models and approximately holomorphic coordinates

We use systematically local charts that *coincide approximately* with the flat model $(\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$ for a 2-calibrated manifolds, where (D_h, J_0) is the natural integrable horizontal distribution $\mathbb{C} \times \{\cdot\}$ with its canonical leafwise complex structure and g_0 the Euclidean metric. Let g_k denote the rescaled metric kg.

Definition 2.1. A family of adapted charts to the sequence (M, D, g_k) is a set of maps $\psi_{k,x} : (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$, $x \in M$, such that $\psi_{k,x}^* D_x = D_h(0)$ and, for k large enough, there are constants $\gamma, \rho_0 > 0$, with $\gamma^{-1} g_0 \leqslant g_k \leqslant \gamma g_0$ such that $|\nabla^j \psi_{k,x}^{-1}|_{g_k} \leqslant \mathrm{O}(1)$ in $B_g(x, \rho_0)$.

Definition 2.2. Let E_k be a sequence of either hermitian or $\mathrm{O}(n)$-vector (or unitary) bundles with compatible connection ∇_k over the almost CR manifold (M, D, J, g) and let $T_k \in \Gamma(E_k)$. We will say that T_k approximately vanishes up to order r, and we will denote it by $T_k \simeq_r 0$, if for every k large enough the inequalities $|\nabla^j T_k|_{g_k} \leqslant \mathrm{O}(k^{-1/2})$, $j = 0, \dots, r$, hold.

If the inequalities hold for all r, we will say that we have an approximated equality and we will denote it simply by $T_k \approxeq 0$.

If $E_k = E$ is a constant sequence, in local adapted coordinates $x_k^1, \ldots, x_k^{2n+1}$ Definition 2.2 requires

$$\left|\partial^{|p|}(\psi_{k,x}^* T_k)/\partial x_k^p\right|_{g_0} \leqslant \mathrm{O}\big(k^{-1/2}\big), \quad p = (p_1, \ldots, p_{2n+1}),\ |p| = p_1 + \cdots + p_{2n+1},\ |p| \leqslant r,$$

at points in $B_{g_k}(0, \mathrm{O}(1))$ and independently of x.

The choice of adapted charts provides trivializations of the bundles associated to TM. Therefore, by choosing a fixed tensor T_0 of the corresponding trivial bundle we are selecting a local tensor for each k and x (for example J_0). In this situation, we say that a given sequence of tensors T_k of the corresponding bundle coincides with T_0 up to order r, if at the points of $B_{g_k}(0, \mathrm{O}(1))$ (and independently of k and x)

$$\left|\partial^{|p|}(\psi_{k,x}^* T_k - T_0)/\partial x_k^p\right|_{g_0} \leqslant \mathrm{O}\big(k^{-1/2}\big), \quad p = (p_1, \ldots, p_{2n+1}),\ |p| = p_1 + \cdots + p_{2n+1},\ |p| \leqslant r.$$

Definition 2.3. We will say that a family $\psi_{k,x} : \mathbb{C}^n \times \mathbb{R} \to U_{k,x} \subset M$ of local adapted charts for (M, D, J, g_k) is approximately holomorphic if $\psi_{k,x}^* D \approxeq D_h$, $V_{k,x}$-defined to be the unique local vector field such that $\partial/\partial s_k + V_k \in \psi_{k,x}^*(D^\perp)$– is of size $\mathrm{O}(1)$ and all its derivatives of size $\mathrm{O}(k^{-1/2})$, and the corresponding coordinate functions $z_k^j : U_{k,x} \to \mathbb{C}$ are approximately holomorphic (see [7]).

Approximately holomorphic coordinates as defined in this note always available. We will assume that we have fixed one such a family. The important property is that a sequence of sections τ_k of E_k is approximately holomorphic (see [7]) if and only if its pullback by the approximately holomorphic charts is a sequence of approximately holomorphic sections of the almost CR manifold $(\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$ (see [8,1] for a more general definition of approximately holomorphic coordinates in the full domain $B_{g_k}(0, O(k^{1/2}))$).

Not chosing an almost CR structure associated to a 2-calibrated manifold, we have the following notion.

Definition 2.4. Let (M, D, ω) be a 2-calibrated manifold. We will say that a local chart $\varphi : \mathbb{C}^n \times \mathbb{R} \to M$ is compatible with ω if $\varphi_*(D_h)$ coincides with D at the origin and, with respect to the canonical complex structure on D_h, the restriction of ω is positive and $(1, 1)$.

3. Proof of the main theorem

We shall denote by E_k as above the sequence of very ample hermitian bundles $\mathbb{C}^2 \otimes L^{\otimes k} \to (M, D, J, g)$. The bundle of pseudo-holomorphic 1-jets along D –$\mathcal{J}_D^1 E_k$– has an approximately holomorphic quasi-stratification with strata Z_k and $\Sigma_{k,n}$ (see [8,7]). The main theorem of [7] asserts the existence of approximately holomorphic sequences of sections τ_k of E_k uniformly transversal along D to these quasi-stratification. Quotienting the components of τ_k we will obtain an approximately holomorphic sequence of maps $\phi_k : M \setminus B_k \to \mathbb{CP}^1$ verifying:

(i) The set $B_k = \tau_k^{-1}(Z_k)$ is a compact submanifold of real codimension 4 uniformly transversal to D.
(ii) The set $\Sigma_n(\phi_k)$ given by the points where $\partial\phi_k$ is singular is a compact submanifold of codimension $2n$ uniformly transversal to D.

The required local model for ϕ_k around the points of B_k follows from the following result.

Proposition 3.1 (see [9,8]). *For every $x \in B_k$ there exist approximately holomorphic coordinates $z_k^1, \ldots, z_k^n, s_k$, centered at x and a holomorphic chart on $\mathbb{CP}^1$ such that in $B_{g_k}(x, \mathrm{O}(1))$, B_k is given by $z_k^1 = z_k^2 = 0$, and $\phi_k(z_k, s_k) = z_k^2/z_k^1$.*

Regarding the points of the singular link, the following result holds (see [8]):

Proposition 3.2. *There exists a perturbation τ'_k of the sequence τ_k with $\tau'_k \approxeq \tau_k$, approximately holomorphic coordinates around any point $y \in \Sigma_n(\phi'_k)$ compatible with ω, and holomorphic charts in $\mathbb{CP}^1$, such that the formula for the maps $\phi'_k = (\tau'_k)^2/(\tau'_k)^1$ is $\phi'_k(z_k, s_k) = \phi'_k(0, s_k) + (z_k^1)^2 + \cdots + (z_k^n)^2$.*

Moreover there are radius $\rho_2 > \rho_1 > 0$, new distributions D_k, almost-complex structures J_k and functions ϕ'_k, such that:

(i) $D_k \approxeq D$ and $J_k \approxeq J$, where $D_k = D$, $J_k = J$ out of a tubular neighborhood of $\Sigma_n(\phi_k)$ or radius ρ_2, and both are integrable in the tubular neighborhood of radius ρ_1.
(ii) $\phi_k \approxeq \phi'_k$, with $\phi_k = \phi'_k$ out of a tubular neighborhood of $\Sigma_n(\phi_k)$ or radius ρ_2, and ϕ'_k holomorphic in the tubular neighborhood of radius ρ_1.

For k large enough, the triple $(A_k, \phi'_k, \Sigma_n(\phi'_k))$ is the Lefschetz pencil structure of Theorem 1.2. The genericity of $\phi'_k(\Sigma_n(\phi'_k))$ is a direct consequence of Proposition 3.2 by considering a perturbation independent of the holomorphic coordinates. Finally an easy adaptation of the argument in [4,9] allows us to modify the function in such a way that we have the desired coordinates.

If the distribution of the 2-calibrated manifold (M, D, J, g) integrates into a foliation $\mathcal{F}$ it is not necessary to choose a retraction r for the projection $p : T^*M \to D^*$ to define higher covariant derivatives along D (and its ∂ and $\bar\partial$ components). We can use on each leaf the Levi-Civita connection defined by $g_{|\mathcal{F}}$. In this way we obtain a natural (foliated) approximately holomorphic theory on M. The approximately holomorphic charts and the charts compatible with ω can be chosen to send $\mathcal{F}$ into the horizontal distribution D_h. In this foliated theory we will obtain again the same estimated transversality theorems and normal forms that were obtained in [7] and [8]. We will attach a subindex $\mathcal{F}$ for objects constructed in the foliated theory. It is easy to show that for E_k very ample, $\tau_k \in \Gamma(E_k)$ is approximately holomorphic if and only it is approximately holomorphic for the foliated theory. Moreover, we have bundle maps $r_j : (D^{*1,0})^{\odot p} \otimes E_k \to (r(D)^{*1,0})^{\odot p} \otimes E_k$, induced by the retraction r associated to the metric. The important observation at this point is that the image of $j^p_{\mathcal{F}}\tau_k$ coincides approximately with $j^p_D\tau_k$ and that the corresponding Thom–Boardmann quasi-stratifications are related by the bundle morphism $\mathcal{J}^p_{\mathcal{F}}E_k \to \mathcal{J}^p_D E_k$ induced by r. Thus, if $j^p_D\tau_k$ is uniformly transversal to the Thom–Boardmann quasi-stratification, the same will happen for $j^p_{\mathcal{F}}\tau_k$. Thus we conclude that the results obtained in [7] and Theorem 1.2 (using the foliated analogs of Propositions 3.1 and 3.2) also hold for foliated almost-complex manifolds and its corresponding foliated almost holomorphic geometry discussed here. In particular, this proves Corollary 1.3.

References

[1] D. Auroux, Estimated transversality in symplectic geometry and projective maps, in: Symplectic Geometry and Mirror Symmetry (Seoul, 2000), World Sci. Publishing, River Edge, NJ, 2001, pp. 1–30.
[2] B. Deroin, Laminations par variétés complexes, Thèse, École Normale Supérieure de Lyon, 2003.
[3] S.K. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Differential Geom. 44 (1996) 666–705.
[4] S.K. Donaldson, Lefschetz fibrations in symplectic geometry, J. Differential Geom. 53 (2) (1999) 205–236.
[5] E. Ghys, Laminations par surfaces de Riemann, in: Dynamique et géométrie complexes (Lyon, 1997), ix, xi, in: Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. 49–95.
[6] E. Giroux, Géométrie de contact : de la dimension trois vers les dimensions supérieures, in: Proceedings of the International Congress of Mathematicians, vol. II, Beijing, 2002, pp. 405–414.
[7] A. Ibort, D. Martínez-Torres, Approximately holomorphic geometry and estimated transversality on 2-calibrated manifolds, C. R. Acad. Sci. Paris, Ser. I. 338 (9) (2004) 709–712.
[8] D. Martínez-Torres, Geometries with topological character, Ph.D. Thesis, Universidad Carlos III de Madrid, 2003.
[9] F. Presas, Lefschetz type pencils on contact manifolds, Asian J. Math. 6 (2) (2002) 277–302.