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When Schröder meets Böttcher – convergence of level sets

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Abstract

It is proven that for families of holomorphic maps with simply connected immediate quadratic basins, the effective level sets of the Schröder or linearizing coordinates converge to the level sets for the Böttcher map, when the multiplier converges to 0. In particular the effective Schröder level sets for $Q_\lambda(z) = \lambda z + z^2$ converge to circles with center 0 as $\lambda \to 0$. ***To cite this article: C.L. Petersen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Quand Schröder rencontre Böttcher. On montre que pour les familles d'applications holomorphes qui ont des bassins immédiats quadratiques et simplement connexes, les ensembles de niveau effectifs de l'application linéarisante de Schröder convergent vers les équipotentielles des coordonnées de Böttcher quand le multiplicateur tend vers zéro. En particulier pour la famille des polynômes quadratiques $Q_\lambda(z) = \lambda z + z^2$ les ensembles de niveau « effectifs » de Schröder convergent vers les cercles centrées en zéro, lorsque $\lambda \to 0$. ***Pour citer cet article : C.L. Petersen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

Let $f : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ be a holomorphic map. Suppose $\alpha \in \Omega$ is an attracting fixed point for f with immediate attracted basin $B_f = B_f(\alpha)$. We shall say that B_f is a simple proper basin if $B_f \simeq \mathbb{D}$ and the restriction $f : B_f \to B_f$ is a proper map. Let $d > 1$ be the degree of the restriction. We call B_f a (simple) quadratic resp. cubic basin if $d = 2$ resp. $d = 3$. In the following we consider only simple proper basins. We denote by $\lambda = f'(0) \in \mathbb{D}$ the multiplier of f at α. For a thorough introduction to the theory of iteration see e.g. Milnors monograph [1].

When $\lambda = 0$ there exists a Böttcher coordinate for f, a univalent map $\phi_f : U \to V$ such that, $\phi_f(\alpha) = 0$ and $\phi_f \circ f = (\phi_f)^k$, where k is the local degree of f at α. The germ of ϕ_f is unique modulo multiplication by a

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$(k-1)$-st. root of unity. The Böttcher potential of f is the subharmonic function $\hat{\kappa}_f(z) = \log|\phi_f(z)|$ extended to B_f by the recursive relation $k \cdot \hat{\kappa}_f(z) = \hat{\kappa}_f(f(z))$. The Böttcher coordinate ϕ_f extends to a biholomorphic map $\phi_f : U_f^1 \to \mathbb{D}(\mathrm{e}^t)$, where either $t = 0$, $U_f^1 = B_f$ and $k = d$ or ∂U_f^1 contains at least one critical point c_f with $\hat{\kappa}_f(c_f) = t$.

When $\lambda \neq 0$ there exists a linearizer or Schröder coordinate for f on B_f. That is a holomorphic map $\phi : B_f \to \mathbb{C}$ with $\phi \circ f = \lambda \cdot \phi$ and $\phi'(\alpha) \neq 0$. Such a map is unique modulo multiplication by a non zero complex number. Let U_f^0 denote the maximal domain which contains α and which is mapped univalently onto a round disk $\mathbb{D}(r)$. Then ∂U_f^0 contains a (possibly several) *first attracted critical point* c with $|\phi(c)| = r$. For a choice of critical point c_f with $\phi(c_f) \neq 0$, we denote by ϕ_f the linearizer with $\phi_f(c_f) = 1$. For c_f a first attracted critical point we denote by $\psi_f : \mathbb{D} \to U_f^0$ the univalent inverse of ϕ_f.

Suppose $f = f_{\mathbf{a}}$ and $\alpha_{\mathbf{a}}$ depend complex analytically on some parameter $\mathbf{a} \in \mathcal{M}$, where $\mathcal{M}$ is some complex analytic manifold. If the local degree $k_{\mathbf{a}}$ at $\alpha_{\mathbf{a}}$ is constant and moreover in case the degree is 1 the critical point $c_f = c_{\mathbf{a}}$ also depends complex analytically on $\mathbf{a}$, then the map $\phi_{\mathbf{a}} = \phi_{f_{\mathbf{a}}}$ (Böttcher or linearizer) depends complex analytically on $(\mathbf{a}, z)$ for $\mathbf{a}$ sufficiently close to $\mathbf{a_0}$.

The level sets for ϕ_f are the sets on which $|\phi_f(z)|$ is constant. These are conveniently discussed via the potential function. For $\lambda \in \mathbb{D}^*$ define $\hat{\kappa}_f : B_f \to [-\infty, \infty[$ by $\hat{\kappa}_f(z) = -(\log|\phi_f(z)|)/\log|\lambda|$, so that $\hat{\kappa}_f(f(z)) = \hat{\kappa}_f(z) - 1$. The function $\hat{\kappa}_f$ is subharmonic with poles at the iterated preimages of α. It is unique modulo an additive constant, which we have fixed so that $\hat{\kappa}_f(c_f) = 0$. The critical points of $\hat{\kappa}_f$ are the critical points c_i of f in B_f and their iterated preimages. The critical values are the numbers $\hat{\kappa}_f(c_i) + n$, $n \in \mathbb{N}$, and includes in particular the non negative integers $\mathbb{N}$.

When a proper basin contains a critical point, $c \neq \alpha$ the level sets are neither nested nor connected. This motivates the notion of essential level sets (defined for both Böttcher and Schröder coordinates): For $t \in \mathbb{R}$ let $U_f(t)$ denote the connected component of $\hat{\kappa}_f^{-1}([-\infty, t[)$ containing 0. Then the sets $U_f(t)$ are nested Jordan domains. The *Milnor filled potential* is the function, defined by

$$\kappa_f(z) = \inf\{s \mid z \in U_f(s)\}.$$

The essential level sets of f are the level sets $K_f(t) = \kappa_f^{-1}(t)$, $t \in \mathbb{R}$ of κ_f. Each (essential) level set has the homotopy type of a circle and bounds $U_f(t)$. It is a Jordan curve iff t is not a critical value for $\hat{\kappa}_f$. It even has interior if t is a critical value. Note that $U_f(t) = \kappa_f^{-1}([-\infty, t[)$. Define *the equilevel set of* $z \in B_f$: $L_f(z) = \kappa_f^{-1}(\kappa_f(z))$.

As a first and principal example consider the quadratic polynomials $Q_\lambda(z) = \lambda z + z^2$, with $\alpha = 0$. To reduce notation we use the index λ synonymously with Q_λ for the above defined entities. We shall in this special case extend the notion of equilevel sets $L_\lambda(z)$ to include the Julia set J_λ as the equilevel set of Julia points and each equipotential set for the Böttcher coordinate at ∞ as the equilevel set of its points.

Let $\mathrm{d}_{\mathbb{C}^*}(\cdot,\cdot)$ denote the complete euclidean metric on $\mathbb{C}^*$ normalized such that $\mathrm{d}_{\mathbb{C}^*}(\mathrm{e}^z, \mathrm{e}^w) \leqslant |z - w|$ with equality iff $\Im(z - w) \leqslant \pi$. Denote by $D_{\mathbb{C}^*}(\cdot,\cdot)$ the Hausdorff distance on the space of compact subsets of $\mathbb{C}^*$ induced by $\mathrm{d}_{\mathbb{C}^*}(\cdot,\cdot)$ and denote by $D(\cdot,\cdot)$ the standard Hausdorff distance between compacts of $\mathbb{C}$. Moreover for $r > 0$ define $C_0(r) = \{z \mid |z| = r\}$.

Theorem 1.1. *For the quadratics* Q_λ *and* $\epsilon > 0$: $D_{\mathbb{C}^*}(L_\lambda(z), C_0(|z|)) \underset{\lambda\to 0}{\longrightarrow} 0$ *uniformly on* $\mathbb{C} \setminus \mathbb{D}(\epsilon)$. *Moreover* $\sup\{D_{\mathbb{C}^*}(L_\lambda(z), C_0(|z|)) | z \in \mathbb{C}^*\} \underset{\lambda\to 0}{\longrightarrow} 2\log(\sqrt{2}+1)$. *In particular* $D(L_\lambda(z), C_0(|z|)) \underset{\lambda\to 0}{\longrightarrow} 0$ *uniformly on* $\mathbb{D}^*$.

Lemma 1.2. *For* Q_λ *and* $\max\{|\lambda|, |\lambda|^{-t}\} < \frac{1}{2}$: $D_{\mathbb{C}^*}(K_\lambda(t), C_0(\frac{1}{4}|\lambda|^{1-t})) \leqslant 6\max\{|\lambda|, |\lambda|^{-t}\}$. *Moreover for any* $0 < t_0 < 1$: $\sup\{D_{\mathbb{C}^*}(L_\lambda(z), C_0(|z|)) | \kappa_\lambda(z) \leqslant t_0\} \underset{|\lambda|\to 0}{\longrightarrow} 2\log(\sqrt{2}+1) < \log 6$.

Proof. We have $K_\lambda(t) = \partial U_\lambda(t) = \psi_\lambda(C_0(|\lambda|^{-t}))$, because in general $z \in \partial K_\lambda(t)$ implies $|\phi_\lambda(z)| = |\lambda|^{-t}$. The first statement is then immediate from the Köbe distortion estimates for univalent maps and the fact that $\psi_\lambda(\lambda) = -\frac{1}{4}\lambda^2$ (the point $-\frac{1}{4}\lambda^2$ is the critical value of Q_λ).

Let $q(z)=2z-z^2$ and for $0<r$ denote by $\delta(r)$ the connected component of $q^{-1}(C_0(r))$ surrounding 0. An easy exercise in Calculus shows that $D_{\mathbb{C}^*}(\delta(r),C_0(|z|))\leqslant 2\log(\sqrt{2}+1)$ for any $z\in\delta(r)$. The restriction $\phi_\lambda:U_\lambda(1)\to\mathbb{D}(|\lambda|^{-1})$ has a unique univalent lift $\theta_\lambda:U_\lambda(1)\to q^{-1}(\mathbb{D}(|\lambda|^{-1}))$ with $\theta_\lambda(0)=0$. The second statement hence follows from the Köbe distortion estimates applied to θ_λ^{-1}, because the conformal modulus $m(U_\lambda(1)\setminus\overline{U_\lambda(t_0)})=(t_0-1)\log|\lambda|/(2\pi)\underset{\lambda\to 0}{\longrightarrow}\infty$. □

Proof of Theorem 1.1. Define $\Sigma_\lambda(t)=\overline{\mathbb{C}}\setminus\kappa_\lambda^{-1}([-\infty,t])$ so that $Q_\lambda:\Sigma_\lambda(t)\to\Sigma_\lambda(t-1)$ is proper of degree 2 and branched only at and above ∞, when $t\geqslant 0$. Define recursively $h_n=h_n^\lambda:\Sigma_\lambda(n)\to\overline{\mathbb{C}}$, $n\geqslant -1$ by $h_{-1}(z)=z$ and $Q_0\circ(h_{n+1}(z))=h_n(Q_\lambda(z))$, with $h_n(z)/z\to 1$ as $z\to\infty$. Then $Q_0^n\circ h_{n-1}=Q_\lambda^n$ on $\Sigma_\lambda(n-1)$. By Lemma 1.2 there exists $0<\delta_0\leqslant 1/6^4$ so small that $|\lambda|\leqslant\delta_0$ implies:

$$\forall t\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right],\ \forall z\in K_\lambda(t):\quad D_{\mathbb{C}^*}\big(K_\lambda(t),C_0\big(|z|\big)\big)\leqslant\log 6\quad\text{and}\tag{1}$$

$$D_{\mathbb{C}^*}\left(K_\lambda\left(-\frac{1}{2}\right),C_0\left(\frac{1}{4}|\lambda|^{3/2}\right)\right)<\log\left(\frac{3}{2}\right).\tag{2}$$

We shall prove that for all $n\geqslant 0$ and for all $z\in\Sigma_\lambda(n+\frac{1}{2})$

$$\mathrm{d}_{\mathbb{C}^*}\big(h_n(z),z\big)\leqslant 6|\lambda|^{1/4}.\tag{3}$$

The theorem is an easy consequence of (3) and (1), because $z\mapsto z^2$ is uniformly infinitesimally expanding with a factor 2 for $\mathrm{d}_{\mathbb{C}^*}$, so that for any compact set $K\subset\mathbb{C}^*$ and $z\in Q_0^{-n}(K)$, $D_{\mathbb{C}^*}(Q_0^{-n}(K),C_0(|z|))=2^{-n}D_{\mathbb{C}^*}(K,C_0(|Q_0^n(z)|))$ and because $[-\frac{1}{2},\frac{1}{2}]$ is a fundamental set of potentials and $\kappa_\lambda(z)\to\infty$ as $|\lambda|\to\infty$ (as Q_λ converges locally uniformly to Q_0 and $K_\lambda(0)\to 0$ by Lemma 1.2).

For $|\lambda|\leqslant\delta_0$ a brief computation shows that $|z|\leqslant\frac{1}{3}|\lambda|^{3/4}$ implies $|Q_\lambda(z)|\leqslant\frac{1}{6}|\lambda|^{3/2}$ so that $Q_\lambda(z)\in U_\lambda(-\frac{1}{2})$ by (2), thus $z\in U_\lambda(\frac{1}{2})$. Hence $\Sigma_\lambda(\frac{1}{2})\subset\overline{\mathbb{C}}\setminus\overline{\mathbb{D}(\frac{1}{3}|\lambda|^{3/4})}$.

For $|z|>|\lambda|$ define $\alpha_0^\lambda(z)=\frac{1}{2}\operatorname{Log}(1+\lambda/z)$ then $h_0(z)=z\exp(\alpha_0^\lambda(z))$ and $\|\alpha_0^\lambda\|\leqslant 3|\lambda|^{1/4}$ on $\Sigma_\lambda(\frac{1}{2})$. Define α_n^λ recursively on $\Sigma_\lambda(n+\frac{1}{2})$ by $\alpha_{n+1}^\lambda(z)=\frac{1}{2}\alpha_n^\lambda(Q_\lambda(z))+\alpha_0^\lambda(z)$. Then by induction $h_n(z)=z\exp(\alpha_n^\lambda(z))$. Moreover likewise by induction $\|\alpha_n^\lambda\|\leqslant 6|\lambda|^{1/4}$ which proves (3). □

Consider complex analytic (parametric) families of holomorphic maps $(\mathbf{b},z)\mapsto f_{\mathbf{b}}(z):\Lambda\times\mathcal{M}\times\Omega\to\mathbb{C}$, where $\mathbf{b}=(\lambda,b)\in\Lambda\times\mathcal{M}$, $f_{\mathbf{b}}(0)=0$ and $f_{\mathbf{b}}'(0)=\lambda$, $\Lambda\subset\mathbb{D}$ and $\Omega\subset\mathbb{C}$ are connected neighbourhoods of the origin and $\mathcal{M}$ is a complex analytic manifold. The family $f_{\mathbf{b}}$ is said to *admit* 0 *as a quadratic fixed point* if the immediate attracted basins $B_{\mathbf{b}}=B_{f_{\mathbf{b}}}(0)$ are all simple quadratic. Examples of such families are abundant among polynomials, e.g. the family Q_λ above and among rational maps, entire and transcendental maps etc. Let $\phi_{\mathbf{b}}:B_{\mathbf{b}}\to\mathbb{D}$ resp. $\mathbb{C}$ denote the Böttcher coordinate for $f_{\mathbf{b}}$, whenever $\lambda=0$ and the Schröder coordinate mapping the unique critical point $c_{\mathbf{b}}$ in $B_{\mathbf{b}}$ to 1, when $\lambda\neq 0$. Then the composite map $\psi_\lambda\circ\phi_{\mathbf{b}}$ is a local conjugacy of $f_{\mathbf{b}}$ to Q_λ and preserves critical values (take $\psi_\lambda=\mathrm{id}$, when $\lambda=0$). It extends to a holomorphic in infact unique biholomorphic conjugacy $\eta_{\mathbf{b}}:B_{\mathbf{b}}\overset{\simeq}{\to}B_\lambda$.

Corollary 1.3 (of Theorem 1.1). *For any complex analytic family $f_{\mathbf{b}}$ admitting* 0 *as a quadratic fixed point*: (i) *The map $(\mathbf{b},z)\mapsto\eta_{\mathbf{b}}(z)$ is complex analytic on $\mathcal{U}=\{(\mathbf{b},z)\mid z\in B_{\mathbf{b}}\}$*, (ii) *the map $\mathbf{b}\mapsto(B_{\mathbf{b}},0)$ is Caratheodory continuous and* (iii) *For any $\mathbf{b_0}=(0,b)$, for any compact subset $K\subset B_{\mathbf{b_0}}\setminus\{0\}$ the Hausdorff distance $D_{\mathbb{C}^*}(L_{\mathbf{b}}(z),L_{\mathbf{b_0}}(z))\underset{\lambda\to 0}{\longrightarrow}0$ uniformly for $z\in K$.*

Proof. Both (i) and (ii) hold when $\lambda\neq 0$, because both $\phi_{\mathbf{b}}$ and ψ_λ are complex analytic. Fix $\mathbf{b_0}=(0,b)$ and let $(K_n)_n$ be an exhaustion of $B_{\mathbf{b_0}}$ (i.e. $\bigcup_n K_n=B_{\mathbf{b_0}}$) with $K_n\simeq\overline{\mathbb{D}}$ and $K_{n-1},f_{\mathbf{b_0}}(K_n)\subset\subset K_n$ for each n. There exist neighbourhoods $\omega_n\subset\Lambda\times\mathcal{M}$ of $\mathbf{b_0}$ such that $K_n\subset B_{\mathbf{b}}$ for every $\mathbf{b}\in\omega_n$. It follows that for each n the set of pointed regions $(B_{\mathbf{b}},0)$, $\mathbf{b}\in\omega_n$ is relatively compact and $U\supset B_{\mathbf{b_0}}$ for any limit point $(U,0)$ of a convergent sequence $(B_{\mathbf{b_n}},0)$, where $\mathbf{b_n}\to\mathbf{b_0}$ as $n\to\infty$. Also the sequence $\eta_{\mathbf{b_n}}:B_{\mathbf{b_n}}\overset{\simeq}{\to}B_{\lambda_n}$ converges to a Riemann map $\hat\eta:U\overset{\simeq}{\to}\mathbb{D}$ of U with $\hat\eta\circ f_{\mathbf{b_0}}=Q_0\circ\hat\eta$. Hence $\hat\eta=\phi_{\mathbf{b_0}}$ and $U=B_{\mathbf{b_0}}$ by uniqueness of Böttcher coordinates. From the continuity of η the rest of the corollary follows. □

2. An application

Consider cubic polynomials $P_{\mathbf{a}}(z)=\lambda z+az^2+z^3$, where $(\lambda,a)=:\mathbf{a}\in\mathbb{C}^2$. Define $\mathcal{H}=\{\mathbf{a}\mid\lambda\in\mathbb{D}$ and $B_{\mathbf{a}}=B_{\mathbf{a}}(0)$ contains both critical points$\}$ and define $\mathcal{H}_0^*=\{(0,a)\in\mathcal{H}\mid a\neq 0\}$.

For $\mathbf{a}\in\mathcal{H}_0^*$ let $\eta_{\mathbf{a}}=\phi_{\mathbf{a}}:U^1_{\mathbf{a}}\to\mathbb{D}(\mathrm{e}^{t_{\mathbf{a}}})$, be Böttcher coordinate with $t_{\mathbf{a}}=\hat\kappa_{\mathbf{a}}(c^1_{\mathbf{a}})$, where $c^1_{\mathbf{a}}\neq 0$ is the second critical point. Similarly for $\mathbf{a}\in\mathcal{H}$ with $\lambda\in\mathbb{D}^*$ let $\phi_{\mathbf{a}}$ be the (a) Schröder coordinate normalized by $\phi_{\mathbf{a}}(c^0_{\mathbf{a}})=1$, where $c^0_{\mathbf{a}}$ is the (a) first attracted critical point. Let $c^1_{\mathbf{a}}$ denote the other critical point and define $t_{\mathbf{a}}=\kappa_{\mathbf{a}}(c^1_{\mathbf{a}})$ and $U^1_{\mathbf{a}}=U_{\mathbf{a}}(t_{\mathbf{a}})$. Suppose $t_{\mathbf{a}}>0$, so that the first attracted critical point is unique. Let $\eta_{\mathbf{a}}:U^1_{\mathbf{a}}\to U_\lambda(t_{\mathbf{a}})$ denote the unique univalent conjugacy between $P_{\mathbf{a}}$ and Q_λ obtained by iterated lifting of the conjugacy $\psi_\lambda\circ\phi_{\mathbf{a}}$. Then as above the map $(\mathbf{a},z)\mapsto\eta_{\mathbf{a}}(z)$ is complex analytic, when $\lambda\neq 0$. Define the equilevel set $L_{\mathbf{a}}(z)$ as in the introduction and set $\mathcal{U}=\{(\mathbf{a},z)\mid z\in U_{\mathbf{a}}$ and either $\mathbf{a}\in\mathcal{H}_0^*$ or $(\lambda\in\mathbb{D}^*\wedge t_{\mathbf{a}}>0)\}$. The following theorem has been applied in the paper [2].

Theorem 2.1. *The map $\eta(\mathbf{a},z)=\eta_{\mathbf{a}}(z)$ is complex analytic on $\mathcal{U}$. In particular for every $\mathbf{a_0}\in\mathcal{H}_0^*$: $\mathbb{D}_{\mathbb{C}^*}(L_{\mathbf{a}}(z),L_{\mathbf{a_0}}(z))\underset{\mathbf{a}\to\mathbf{a_0}}{\longrightarrow}0$ uniformly on compact subsets of $B_{\mathbf{a_0}}\setminus\{0\}$.*

Proof. For $U\subset\mathbb{C}$ an open subset containing 0 define $r(U)=\sup\{r\mid\mathbb{D}(r)\subset U\}$. The proof of the following claim is an easy corollary of Theorem 1.1 and is left to the reader:

Claim. *For every $r_0\in\,]0,1[$ there exists $\delta>0$ such that for all $|\lambda|<\delta$ and every $t\in\mathbb{R}$ with $r(U_\lambda(t))\leqslant r_0$ the subset $U_\lambda(t-\frac12)$ is contained in $D_{U_\lambda(t)}(0,2\log\frac{1+\sqrt r}{1-\sqrt r})$ the hyperbolic ball in $U_\lambda(t)$ with center 0 and radius $2\log\frac{1+\sqrt r}{1-\sqrt r}$.*

A simple calculation shows that $\eta'_{\mathbf{a}}(0)\to a_0$ as $\mathbf{a}\to\mathbf{a_0}\in\mathcal{H}_0^*$. Fix $\mathbf{a_0}\in\mathcal{H}_0^*$, then there exists $r_0>0$ and $\delta>0$ such that $r(U^1_{\mathbf{a}})>r_0$ for all $|\mathbf{a}-\mathbf{a_0}|<\delta$. Suppose to the contrary that $r(U^1_{\mathbf{a_n}})\to 0$ for some sequence $\mathbf{a_n}\to\mathbf{a_0}$. Then also $r(U_\lambda(t_{\mathbf{a}}))\to 0$, by the Köbe $\frac14$-theorem. But then $\overline{U_{\mathbf{a}}(t_{\mathbf{a}}-\frac12)}$, which contains $v^1_{\mathbf{a}}$ converges Hausdorff to $\{0\}$ by the claim, this contradicts that $v^1_{\mathbf{a}}\to v^1_{\mathbf{a_0}}\neq 0$. Hence the pointed regions $(U^1_{\mathbf{a}},0)$ are precompact for the Caratheodory topology. Let $(\mathbf{a_n})$ be a sequence converging to $\mathbf{a_0}$. Passing to a subsequence we can suppose that $(U^1_{\mathbf{a}},0)$ converges Caratheodory to a pointed region $(U,0)$ and that the conjugacies $\eta_{\mathbf{a_n}}:U^1_{\mathbf{a_n}}\to U_{\lambda_n}(t_{\mathbf{a_n}})$ converges locally uniformly to $\hat\eta_{\mathbf{a_0}}:U\to\mathbb{D}(\mathrm{e}^t)$ a uniformizing map which conjugates $P_{\mathbf{a_0}}$ to Q_0. Hence $\hat\eta_{\mathbf{a_0}}=\phi_{\mathbf{a_0}}=\eta_{\mathbf{a_0}}$ on U and $\eta_{\mathbf{a_0}}$ is continuous on $\{\mathbf{a_0}\}\times U$. Also $t<t^1_{\mathbf{a_0}}=\kappa_{\mathbf{a_0}}(c^1_{\mathbf{a_0}})$, because $\hat\eta_{\mathbf{a_0}}$ is univalent, so that $U\subset U^1_{\mathbf{a_0}}$. Finally $t=t^1_{\mathbf{a_0}}$ as $|\eta_{\mathbf{a_n}}(P_{\mathbf{a_n}}(c^1_{\mathbf{a_n}}))|=|\eta_{\mathbf{a_n}}(v^1_{\mathbf{a_n}})|\to\mathrm{e}^{2t}$, so that $U=U^1_{\mathbf{a_0}}$. From the continuity the theorem follows. □

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