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Partial Differential Equations

A new concept of reduced measure for nonlinear elliptic equations

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Abstract

We study the existence of solutions of the nonlinear problem

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{i}$$

where μ is a Radon measure and $g:\mathbb{R}\to\mathbb{R}$ is a nondecreasing continuous function with $g(t)=0$, $\forall t\leqslant 0$. Given g, Eq. (i) need not have a solution for every measure μ, and we say that μ is a good measure if (i) admits a solution. We show that for every μ there exists a largest good measure $\mu^*\leqslant\mu$. This *reduced measure* μ^* has a number of remarkable properties. ***To cite this article: H. Brezis et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Un nouveau concept de mesure réduite pour des équations elliptiques non linéaires. On étudie l'existence de solutions du problème non linéaire

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{ii}$$

où μ est une mesure de Radon et g est une fonction croissante et continue avec $g(t)=0$, $\forall t\leqslant 0$. Étant donné g, l'Éq. (ii) n'admet pas nécessairement de solution pour toute mesure μ. On dit que μ est une bonne mesure (relative à g) si (ii) admet une solution. On démontre que pour toute mesure μ, il existe une plus grande bonne mesure $\mu^*\leqslant\mu$. La *mesure réduite* μ^* a plusieurs propriétés remarquables. ***Pour citer cet article : H. Brezis et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Soit $\Omega \subset \mathbb{R}^N$ un domaine borné régulier. Soit $g:\mathbb{R} \to \mathbb{R}$ une fonction croissante et continue telle que $g(t) = 0$ pour tout $t \leqslant 0$. On s'intéresse au problème

$$\begin{cases} -\Delta u + g(u) = \mu & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1}$$

où μ est une mesure de masse totale finie sur Ω.

Étant donnés g et μ, l'Éq. (1) n'admet pas nécessairement de solution. Pour g *fixé*, on dit que $\mu \in \mathcal{M}(\Omega)$ est une *bonne mesure* (relative a g) si (1) admet une solution. On désigne par $\mathcal{G}$ l'ensemble des bonnes mesures associées à la non linéarité g.

Soit $g_n(t) = \min\{g(t), n\}$, $\forall t \in \mathbb{R}$. Dans ce cas, pour tout $n \geqslant 1$, le problème

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & \text{dans } \Omega, \\ u_n = 0 & \text{sur } \partial\Omega, \end{cases} \tag{2}$$

admet une unique solution $u_n \in L^1(\Omega)$. Le comportement de la suite (u_n) lorsque $n \to \infty$ est donné par :

Proposition 0.1. *Soit u_n l'unique solution de* (2). *Alors, $u_n \downarrow u^*$ lorsque $n \uparrow \infty$, où u^* est la plus grande sous-solution de* (1). *De plus,*

$$\left|\int_\Omega u^*\Delta\zeta\right| \leqslant 2\|\mu\|_{\mathcal{M}}\|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}) \quad \textit{et} \quad \int_\Omega g(u^*) \leqslant \|\mu\|_{\mathcal{M}}. \tag{3}$$

De (3) on déduit qu'il existe une unique mesure μ^*, appelée *mesure réduite*, telle que

$$-\int_\Omega u^*\Delta\zeta + \int_\Omega g(u^*)\zeta = \int_\Omega \zeta\,\mathrm{d}\mu^* \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{4}$$

Par définition, la mesure réduite μ^* est une bonne mesure. De plus, comme u^* est une sous-solution de (1), on a $\mu^* \leqslant \mu$.

Voici quelques-uns de nos résultats principaux :

Théorème 0.2. *La mesure réduite μ^* est la plus grande bonne mesure $\leqslant \mu$.*

Théorème 0.3. *Il existe un borélien $\Sigma \subset \Omega$, avec* $\operatorname{cap}(\Sigma) = 0$, *tel que $(\mu - \mu^*)(\Omega \setminus \Sigma) = 0$, où* « cap » *désigne la capacité newtonienne.*

Corollaire 0.4. *Si $\mu_1, \mu_2 \in \mathcal{G}$, alors* $\sup\{\mu_1, \mu_2\} \in \mathcal{G}$.

Corollaire 0.5. *$\mathcal{G}$ est convexe.*

Corollaire 0.6. *Pour toute mesure μ, on a $\|\mu - \mu^*\|_{\mathcal{M}} = \min_{\nu \in \mathcal{G}} \|\mu - \nu\|_{\mathcal{M}}$, c'est-à-dire μ^* est la meilleure approximation de μ dans $\mathcal{G}$.*

Théorème 0.7. *Soit $\mu \in \mathcal{M}(\Omega)$. Alors, μ est une bonne mesure pour tout g si et seulement si $\mu^+(A) = 0$ pour tout borélien $A \subset \Omega$ tel que* $\operatorname{cap}(A) = 0$.

Les démonstrations détaillées sont présentées dans [7].

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Let $g:\mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function such that $g(0) = 0$. In this paper we are concerned with the problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where μ is a measure. It is well-known (see [8]) that, for every $\mu \in L^1(\Omega)$, problem (5) admits a unique weak solution. The right concept of weak solution is the following:

$$u \in L^1(\Omega), \quad g(u) \in L^1(\Omega) \quad \text{and} \quad -\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = \int_\Omega \zeta\,\mathrm{d}\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \tag{6}$$

where $C_0^2(\overline{\Omega}) = \{\zeta \in C^2(\overline{\Omega});\ \zeta = 0 \text{ on } \partial\Omega\}$.

The case where μ is a measure turns out to be much more subtle than one might expect. It was observed in 1975 by Bénilan and Brezis (see [2] and also [4]) that if $N \geqslant 3$ and $g(t) = |t|^{p-1}t$ with $p \geqslant \frac{N}{N-2}$, then (6) has *no solution* when $\mu = \delta_a$, a Dirac mass at a point $a \in \Omega$. On the other hand, it was also proved that if $g(t) = |t|^{p-1}t$ with $p < \frac{N}{N-2}$ (and $N \geqslant 2$), then (6) has a solution for any measure μ.

Our goal in this Note is to analyze the nonexistence mechanism and to describe what happens if one attempts to approximate a solution of (5) in cases where the equation does not possess a solution. We apply several approximation schemes. For example, μ is kept fixed and g is truncated. Alternatively, g is kept fixed and μ is approximated, e.g., via convolution. If $N \geqslant 3$, $g(t) = |t|^{p-1}t$, with $p \geqslant \frac{N}{N-2}$, and $\mu = \delta_a$, with $a \in \Omega$, then all 'natural' approximations (u_n) of (5) converge to $u \equiv 0$ (see [5]). And, of course, $u \equiv 0$ is *not* a solution of (5) corresponding to $\mu = \delta_a$! It is this kind of phenomenon that we propose to explore in full generality. We are led to study the convergence of the approximate solutions (u_n) for various approximation schemes.

Concerning the function g we will assume that $g:\mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and that $g(t) = 0$, $\forall t \leqslant 0$. This last assumption is harmless when the data μ is nonnegative, since the corresponding solution u is nonnegative by the maximum principle and it is only the restriction of g to $[0, \infty)$ which is relevant.

Let $\mathcal{M}(\Omega)$ denote the space of finite measures μ on $\overline{\Omega}$ with $|\mu|(\partial\Omega) = 0$. By a (weak) *solution* u of (5) we mean that (6) holds for some given $\mu \in \mathcal{M}(\Omega)$. A (weak) *subsolution* u of (5) is a function u satisfying

$$u \in L^1(\Omega), \quad g(u) \in L^1(\Omega) \quad \text{and} \quad -\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta \leqslant \int_\Omega \zeta\,\mathrm{d}\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geqslant 0 \text{ in } \Omega. \tag{7}$$

We say that $\mu \in \mathcal{M}(\Omega)$ is a *good measure* if (5) admits a solution. If μ is a good measure, then equation (5) has exactly one solution u. We denote by $\mathcal{G}$ the set of good measures (relative to g).

In the sequel, we will introduce the first approximation method, namely μ is fixed and g is 'truncated'. Let (g_n) be a sequence of bounded functions $g_n:\mathbb{R} \to \mathbb{R}$, which are continuous, nondecreasing and satisfy the following conditions:

$$0 \leqslant g_1(t) \leqslant g_2(t) \leqslant \cdots \leqslant g(t) \quad \text{and} \quad g_n(t) \to g(t) \quad \forall t \in \mathbb{R}. \tag{8}$$

A good example to keep in mind is $g_n(t) = \min\{g(t), n\}$, $\forall t \in \mathbb{R}$.

Our first result is

Proposition 1.1. *Given any measure $\mu \in \mathcal{M}(\Omega)$, let u_n be the unique solution of*

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Then $u_n \downarrow u^*$ in Ω as $n \uparrow \infty$, where u^* is the largest subsolution of (5). *Moreover we have*

$$\left| \int_\Omega u^* \Delta\zeta \right| \leqslant 2\|\mu\|_{\mathcal{M}} \|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}) \quad \textit{and} \quad \int_\Omega g(u^*) \leqslant \|\mu\|_{\mathcal{M}}. \tag{10}$$

An important consequence of Proposition 1.1 is that u^* *does not depend on the choice of the truncating sequence* (g_n). *It is an intrinsic object which will play an important role in the sequel*. In some sense, u^* is the 'best one can do'(!) in the absence of a solution. Note that if μ is a good measure, then u^* coincides with the unique solution u of (5).

From (10) we see that there exists a unique measure $\mu^* \in \mathcal{M}(\Omega)$ such that

$$-\int_\Omega u^* \Delta\zeta + \int_\Omega g(u^*)\zeta = \int_\Omega \zeta \, \mathrm{d}\mu^* \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{11}$$

We call μ^* the *reduced measure* associated to μ. Clearly, μ^* is always a good measure. Since u^* is a subsolution of (5), we have $\mu^* \leqslant \mu$. Even though we have not indicated the dependence on g we emphasize that μ^* *does depend* on g.

One of our main results is

Theorem 1.2. *The reduced measure μ^* is the largest good measure $\leqslant \mu$.*

Two main ingredients in the proof of Theorem 1.2 are the 'Inverse' maximum principle (see [9]) and Kato's inequality when Δu is a measure (see [6]).

Here is an easy consequence of Theorem 1.2:

Corollary 1.3. *We have* $0 \leqslant \mu - \mu^* \leqslant \mu^+ = \sup\{\mu, 0\}$. *In particular,* $|\mu^*| \leqslant |\mu|$; *moreover, if* $\mu \geqslant 0$, *then* $\mu^* \geqslant 0$.

Our next result asserts that the measure $\mu - \mu^*$ is concentrated on a small set:

Theorem 1.4. *There exists a Borel set $\Sigma \subset \Omega$ with* $\operatorname{cap}(\Sigma) = 0$ *such that* $(\mu - \mu^*)(\Omega \setminus \Sigma) = 0$.

Here and throughout the rest of the paper 'cap' denotes the Newtonian (H^1) capacity with respect to Ω.

Remark 1. Theorem 1.4 is optimal in the following sense. Given any measure $\mu \geqslant 0$ concentrated on a set of zero capacity, there exists some g such that $\mu^* = 0$. In particular, $\mu - \mu^*$ can be *any* nonnegative measure concentrated on a set of zero capacity.

A measure $\mu \in \mathcal{M}(\Omega)$ is called *diffuse* if $|\mu|(A) = 0$ for every Borel set $A \subset \Omega$ such that $\operatorname{cap}(A) = 0$. We shall denote by $\mathcal{M}_{\mathrm{d}}(\Omega)$ the set of diffuse measures. It has been known (see [3]) that a measure μ is diffuse if and only if $\mu = f - \Delta v$ for some $f \in L^1(\Omega)$ and $v \in H_0^1(\Omega)$; a sharper version (see [7]) asserts that one may even choose v such that $v \in C(\overline{\Omega}) \cap H_0^1(\Omega)$.

An immediate consequence of Corollary 1.3 and Theorem 1.4 is

Corollary 1.5. *Every diffuse measure μ is a good measure.*

Remark 2. The converse of Corollary 1.5 is *not* true. If $N = 2$ and $g(t) = \mathrm{e}^t - 1$, $t \geqslant 0$, then the measure $\mu = c\delta_a$, with $0 < c \leqslant 4\pi$ and $a \in \Omega$, is a good measure, but it is not diffuse.

Here are some basic properties of the good measures:

Theorem 1.6. *Suppose μ_1 is a good measure. Then any measure $\mu_2 \leqslant \mu_1$ is also a good measure.*

We now deduce a number of consequences:

Corollary 1.7. *Let $\mu \in \mathcal{M}(\Omega)$. If μ^+ is diffuse, then μ is a good measure.*

Corollary 1.8. *If μ_1 and μ_2 are good measures, then so is $\nu = \sup\{\mu_1, \mu_2\}$.*

Corollary 1.9. *$\mathcal{G}$ is convex.*

Corollary 1.10. *For every measure $\mu \in \mathcal{M}(\Omega)$ we have $\|\mu - \mu^*\|_{\mathcal{M}} = \min_{\nu\in\mathcal{G}} \|\mu - \nu\|_{\mathcal{M}}$, so that the reduced measure μ^* is the best approximation of μ in $\mathcal{G}$.*

As we have already pointed out, the set of good measures $\mathcal{G}$ associated to (5) depends on the nonlinearity g. By Corollary 1.7, if $\mu \in \mathcal{M}(\Omega)$ and μ^+ is diffuse, then μ is a good measure for every g. The converse is also true; more precisely,

Theorem 1.11. *A measure μ is good for every g if and only if μ^+ is diffuse.*

We also have the following

Theorem 1.12. *A measure $\mu \in \mathcal{M}(\Omega)$ is a good measure if and only if μ admits a decomposition*

$$\mu = f_0 - \Delta v_0 \quad \text{in } \left[C_0^2(\overline{\Omega})\right]^*, \tag{12}$$

with $f_0 \in L^1(\Omega)$, $v_0 \in L^1(\Omega)$ and $g(v_0) \in L^1(\Omega)$.

When $g(t) = t^p$, $t \geqslant 0$, this result is due to Baras–Pierre (see [1]). Next,

Theorem 1.13. *We have $\mathcal{G} + \mathcal{M}_{\rm d}(\Omega) \subset \mathcal{G}$.*

Here are some basic properties of the mapping $\mu \mapsto \mu^*$:

Theorem 1.14. *For every $\mu, \nu \in \mathcal{M}(\Omega)$, $|\mu^* - \nu^*| \leqslant |\mu - \nu|$. Moreover, if $\mu \leqslant \nu$, then $\mu^* \leqslant \nu^*$.*

Here are some examples where μ^* can be explicitly computed in terms of μ:

Example 1. Assume $N \geqslant 2$ and $g(t) = t^p$, $t \geqslant 0$, for some $1 < p < \infty$. If $1 < p < \frac{N}{N-2}$, then by a result of Bénilan–Brezis (see [2]) problem (5) has a solution for every measure μ; thus, $\mu^* = \mu$. If $p \geqslant \frac{N}{N-2}$, then using a result of Baras–Pierre (see [1]) it is possible to show that $\mu^* = \mu - (\mu_2)^+$, where μ_2 denotes the part of μ which is concentrated on a set of zero $W^{2,p'}$-capacity.

Example 2. Assume $N = 2$ and $g(t) = \mathrm{e}^t - 1$, $t \geqslant 0$. Using a result of Vázquez (see [10]), one can prove that $\mu^* = \mu_1 + \sum_i \min\{\alpha_i, 4\pi\}\delta_{a_i}$, where μ_1 denotes the non-atomic part of μ and $\sum_i \alpha_i \delta_{a_i} = \mu - \mu_1$ denotes its atomic part.

Open problem 1. Let $N = 2$ and $g(t) = (\mathrm{e}^{t^2} - 1)$, $t \geqslant 0$. Is there an explicit formula for μ^*?

Open problem 2. Let $N \geqslant 3$ and $g(t) = (\mathrm{e}^t - 1)$, $t \geqslant 0$. Is there an explicit formula for μ^*?

Another approximation scheme is the following. We now keep g fixed but we smooth μ via convolution. More precisely, given a sequence (ρ_n) of mollifiers in $\mathbb{R}^N$ such that supp $\rho_n \subset B_{1/n}$ for every $n \geqslant 1$, set $\mu_n = \rho_n * \mu$. Let u_n be the solution of (5) with μ_n instead of μ.

Theorem 1.15. *Assume in addition g is convex. Then $u_n \to u^*$ in $L^1(\Omega)$, where u^* is given by Proposition* 1.1.

We conclude with the following

Open problem 3. Does the conclusion of Theorem 1.15 remain valid without the convexity assumption on g?

Detailed proofs will appear in [7].

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