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Probability Theory

The strong solution of the Monge–Ampère equation on the Wiener space for log-concave densities

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Abstract

Let (W, H, μ) be an abstract Wiener space, assume that $\mathrm{d}\nu = L\,\mathrm{d}\mu$ is a second probability measures on $(W, \mathcal{B}(W))$ such that $L = \frac{1}{c}\exp -f$, with $f \in \mathbb{D}_{2,1}$ lower bounded and H-convex. Let $T = I_W + \nabla\varphi$, $\varphi \in \mathbb{D}_{2,1}$, be the solution of the Monge problem transporting μ to ν and realizing the H-Wasserstein distance between μ and ν. We prove that $\varphi \in \mathbb{D}_{2,2}$ hence the Gaussian Jacobian $\Lambda = \det_2(I + \nabla^2\varphi)\exp\{\mathcal{L}\varphi - 1/2|\nabla\varphi|_H^2\}$ is well-defined and T is the strong solution of the Monge–Ampère equation $\Lambda L \circ T = 1$ a.s. on W. ***To cite this article: D. Feyel, A.S. Üstünel, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

La solution forte de l'équation de Monge–Ampère sur l'espace de Wiener pour les densités log-concaves. Soit (W, H, μ) un espace de Wiener abstrait, on suppose que $\mathrm{d}\nu = L\,\mathrm{d}\mu$ est une autre probabilité sur $(W, \mathcal{B}(W))$ où $L = \frac{1}{c}\exp -f$, avec $f \in \mathbb{D}_{2,1}$, inférieurement bornée et H-convexe. Soit $T = I_W + \nabla\varphi$, $\varphi \in \mathbb{D}_{2,1}$, la solution du problème de Monge qui transporte μ sur ν et qui realise la distance de Wasserstein entre μ et ν par rapport à la métrique de Cameron–Martin. Nous montrons qu'en fait $\varphi \in \mathbb{D}_{2,2}$. Par conséquent le jacobien gaussien $\Lambda = \det_2(I + \nabla^2\varphi)\exp\{\mathcal{L}\varphi - 1/2|\nabla\varphi|_H^2\}$ est bien défini et T est la solution forte de l'equation de Monge–Ampère $\Lambda L \circ T = 1$ p.s. ***Pour citer cet article : D. Feyel, A.S. Üstünel, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit (W, H, μ) un espace de Wiener abstrait : W est un Fréchet séparable localement convexe, μ est une mesure gaussienne dont le support est W et H est l'espace de Cameron–Martin dont le produit scalaire et la norme sont notés respectivement $(\cdot,\cdot)_H$ et $|\cdot|_H$. On notera par ∇ la fermeture par rapport à μ de la dérivée dans la direction

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de H. En particulier, pour un espace hilbertien M, $\mathbb{D}_{2,k}(M)$ est l'espace de classes d'équivalences de fonctions mesurables, à valeurs dans M, dont les dérivées d'ordre $k \in \mathbb{N}$ sont de carré intégrables par rapport à la norme du produit tensoriel Hilbert–Schmidt $M \otimes H^{\otimes k}$, où $H^{\otimes k}$ est l'espace des k-tenseurs Hilbert–Schmidt ; si $M = \mathbb{R}$ alors nous noterons $\mathbb{D}_{2,k}$ au lieu de $\mathbb{D}_{2,k}(\mathbb{R})$ (cf. [7,12,13]). On notera par δ l'adjoint de ∇ par rapport à μ, qui est une application continue de $\mathbb{D}_{2,1}(M \otimes H^{\otimes k+1})$ dans $\mathbb{D}_{2,1}(M \otimes H^{\otimes k})$. Rappelons que $\delta \circ \nabla$ est l'opérateur d'Ornstein–Uhlenbeck, il sera noté $\mathcal{L}$.

Soit ν une autre probabilité, notons par $\Sigma(\mu, \nu)$ l'ensemble des probabilités sur $W \times W$ de marginales μ et ν. On note J la fonctionnelle définie sur $\Sigma(\mu, \nu)$ par $J(\beta) = \int_{W \times W} |x - y|_H^2 \, d\beta(x, y)$. Dans le cas où W est de dimension finie, le problème de Monge–Kantorovitch consiste à trouver une mesure $\gamma \in \Sigma(\mu, \nu)$ telle que la distance de Wasserstein

$$d_H^2(\mu, \nu) = \inf\{J(\beta) \colon \beta \in \Sigma(\mu, \nu)\}$$

soit atteinte en γ. Ce problème a été résolu dans [2] en dimension finie (cf. aussi [6] pour un survol rapide). Nous l'avons résolu dans [9,10] (cf. aussi [11]) quand la dimension de H est infinie. Expliquons plus précisément le cas particulier qui sera utilisé dans cette Note : si ν est de la forme $d\nu = L\,d\mu$, alors il existe une fonction φ, appelée le potentiel de transport, appartenant à $\mathbb{D}_{2,1}$, telle que $T : W \to W$ définie par $T = I_W + \nabla\varphi$ satisfasse $T\mu = \nu$ et telle que $\gamma = (I_W \times T)\mu$ soit l'unique mesure dans $\Sigma(\mu, \nu)$ satisfaisant $J(\gamma) = d_H^2(\mu, \nu)$. De plus φ est 1-convexe : une variable aléatoire $f : W \to \mathbb{R} \cup \{\infty\}$ est dite r-convexe, $r \in \mathbb{R}$, si $h \to \frac{r}{2}|h|_H^2 + f(w + h)$ est convexe sur H à valeurs dans $\mathbb{L}^0(\mu)$ [8] ; si $r = 0$, on l'appelle H-convexe. Avec les hypothèses ci-dessus T admet un inverse p.s., noté S, de la forme $S = I_W + \eta$. De plus si ∇ est fermable par rapport à ν alors $\eta : W \to H$ est de la forme $\eta = \nabla\psi$ où $\psi \in L^2(\nu)$ est ν-differentiable dans la direction de H. Dans le cas qui nous intéresse, il est important de savoir si φ est un élément de $\mathbb{D}_{2,2}$ au lieu de $\mathbb{D}_{2,1}$, pour pouvoir calculer le jacobien

$$\Lambda = \det{}_2(I_H + \nabla^2\varphi) \exp\left\{-\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|_H^2\right\},$$

où $\det_2(I_H + \nabla^2\varphi)$ est le déterminant modifié de Carleman–Fredholm (cf. [5,14]).

1. Main results

Here is the first notable result of this Note:

Theorem 1.1. *Assume that ν is of the form $d\nu = \frac{1}{c} e^{-f} d\mu$, where c is the normalization constant, $f \in \mathbb{D}_{2,1}$ is H-convex, lower bounded, i.e., $f \geqslant -\alpha$ a.s., for some $\alpha > 0$. Then the transport potential φ belongs to the second order Sobolev space $\mathbb{D}_{2,2}$.*

For the proof we need the following lemma whose proof, in a much more general situation, can be found in [14], Appendix B:

Lemma 1.2. *Assume that $N : W \to W$ is a map of the form $N = I_W + u$, where $u \in \mathbb{D}_{2,1}(H)$ such that the image measure $N\mu$ is absolutely continuous with respect to μ. For any smooth, cylindrical map $\xi : W \to H$, we have*

$$\delta\xi \circ N = \delta(\xi \circ N) + (\xi \circ N, u)_H + \operatorname{trace}(\nabla\xi \circ N \cdot \nabla u).$$

Proof of Theorem 1.1. Let $(e_n, n \geqslant 1)$ be a CONB of H, denote by V_n the sigma algebra on W generated by $\{\delta e_1, \ldots, \delta e_n\}$ and let $L_n = E[P_{1/n} L | V_n]$, where the conditional expectation is with respect to the Wiener measure μ and $P_{1/n}$ denotes the Ornstein–Uhlenbeck semigroup $e^{-t\mathcal{L}}$ at $t = 1/n$. From [8], L_n is of the form $\frac{1}{c} e^{-f_n}$, where f_n is an H-convex function of the form $\tilde{f}_n(\delta e_1, \ldots, \delta e_n)$ such that $\tilde{f}_n$ is a smooth, convex function on $\mathbb{R}^n$. From the results of Caffarelli [3,4], the transport potentials φ_n and ψ_n associated to the measures (μ, ν_n), where $\mathrm{d}\nu_n = L_n\,\mathrm{d}\mu$, are $\mathcal{C}^2$-functions. Moreover $\nabla\varphi_n$ and $T_n = I_W + \nabla\varphi_n$ are 1-Lipschitz maps, i.e., $|\nabla\varphi_n(x+h) - \nabla\varphi_n(x)|_H \leqslant |h|_H$ and $|T_n(x+h) - T_n(x)|_H \leqslant |h|_H$ for any $x \in W,\ h \in H$. Consequently $\phi_n \in \mathbb{D}_{2,2}$. Besides, the Lipschitz property combined with the Poincaré inequality, cf. [13] or the Appendix of [14], implies that

$$\sup_n E\big[\exp \varepsilon |\nabla\varphi_n|_H^2\big] < \infty, \tag{1}$$

for some $\varepsilon > 0$. Since $\delta \circ \nabla = \mathcal{L}$ and since the dual transport potential is C^2, we have, using Lemma 1.2

$$\mathcal{L}\psi_n \circ T_n = \delta(\nabla\psi_n \circ T_n) + (\nabla\psi_n \circ T_n, \nabla\varphi_n)_H + \operatorname{trace}\big(\nabla^2\psi_n \circ T_n \cdot \nabla^2\varphi_n\big). \tag{2}$$

Note that L_n is strictly positive, hence $I_H + \nabla^2\varphi_n(x)$ is invertible for μ-almost all $x \in W$. Besides, it is easy to see that

$$\operatorname{trace}\big(\nabla^2\psi_n \circ T_n \cdot \nabla^2\varphi_n\big) = -\operatorname{trace}\big((I_H + \nabla^2\varphi_n)^{-1} \cdot (\nabla^2\varphi_n)^2\big).$$

Inserting this relation in (2) and taking the expectation of both sides w.r.to μ gives

$$\begin{aligned} E\big[\operatorname{trace}\big((I_H + \nabla^2\varphi_n)^{-1} \cdot (\nabla^2\varphi_n)^2\big)\big] &= -E\big[(\nabla\psi_n \circ T_n, \nabla\varphi_n)_H\big] - E[\mathcal{L}\psi_n \circ T_n] \\ &= E\big[|\nabla\varphi_n|_H^2\big] - E[\mathcal{L}\psi_n\, L_n]. \end{aligned} \tag{3}$$

In the equality (3), we have used the fact that $\nabla\psi_n \circ T_n = -\nabla\varphi_n$ and that $T_n\,\mathrm{d}\mu = L_n\,\mathrm{d}\mu$. Since $(L_n, n \geqslant 1)$ is bounded in $L^\infty(\mu)$ by some K, we have

$$E[\mathcal{L}\psi_n\ L_n] = E\big[(\nabla\psi_n, \nabla L_n)_H\big] = -E\big[(\nabla\psi_n, \nabla f_n)_H\, L_n\big] \leqslant K\|\nabla\psi_n\|_{L^2(\mu,H)}\|f\|_{2,1},$$

where $\|f\|_{2,1} = \{E[|\nabla f|_H^2 + |f|^2]\}^{1/2}$ is the norm of $\mathbb{D}_{2,1}$. Note that, from the finite dimensional Jacobi theorem, we have $L_n \circ T_n\, \Lambda_n = 1$ a.s., where $\Lambda_n = \det_2(I_H + \nabla^2\varphi_n)\exp(-\mathcal{L}\varphi_n - (1/2)|\nabla\varphi_n|_H^2)$. Hence $S_n(\mathrm{d}\mu) = \Lambda_n\,\mathrm{d}\mu$, where $S_n = I_W + \nabla\psi_n$ is the inverse of T_n. Moreover, from the Young inequality

$$\begin{aligned} E\big[|\nabla\psi_n|_H^2\big] &= E\big[|\nabla\varphi_n \circ S_n|_H^2\big] \\ &= E\big[|\nabla\varphi_n|_H^2\,\Lambda_n\big] \\ &\leqslant \frac{1}{\varepsilon} E[\Lambda_n \log \Lambda_n] + E\big[\exp \varepsilon|\nabla\varphi_n|_H^2\big], \end{aligned}$$

which is uniformly bounded w.r.to n by (1) and by the fact that $E[\Lambda_n \log \Lambda_n] = E[-\log L_n] = \log c + E[f_n] \leqslant \log c + \|f\|_{2,1}$. Consequently

$$\sup_n E\big[\operatorname{trace}\big((I_H + \nabla^2\varphi_n)^{-1} \cdot (\nabla^2\varphi_n)^2\big)\big] < \infty. \tag{4}$$

Recalling that the operator norm $\|I_H + \nabla^2\varphi_n\|_{\mathrm{op}} \leqslant 1$, we finally get

$$\begin{aligned} \sup_n E\big[\|\nabla^2\varphi_n\|_2^2\big] &= \sup_n E\big[\operatorname{trace}(\nabla^2\varphi_n)^2\big] \\ &\leqslant \sup_n E\big[\big\|(I_H + \nabla^2\varphi_n)^{-1/2}\nabla^2\varphi_n\big\|_2^2\big] \\ &= \sup_n E\big[\operatorname{trace}\big((I_H + \nabla^2\varphi_n)^{-1}(\nabla^2\varphi_n)^2\big)\big] < \infty, \end{aligned}$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. This result implies that the sequence $(\varphi_n,\ n \geqslant 1)$ is bounded in $\mathbb{D}_{2,2}$, since it converges to φ in $\mathbb{D}_{2,1}$, cf. [10], φ should be an element of $\mathbb{D}_{2,2}$. □

The following corollary follows from Theorem 1.1 and the Fatou lemma:

Corollary 1.3. *Let Λ be the function defined by*

$$\Lambda = \det_2(I_H + \nabla^2\varphi)\exp\left[-\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|_H^2\right].$$

Then T is a subsolution of the Monge–Ampère equation in the sense that

$$\Lambda \leqslant \frac{1}{L\circ T}$$

almost surely.

Now we can prove the main result of this Note:

Theorem 1.4. *T is the strong solution of the Monge–Ampère equation, in other words*

$$\Lambda = \frac{1}{L\circ T}$$

almost surely.

Proof. Let $(\phi_n',\ n\geqslant 1)$ be the sequence such that ϕ_n' is constructed from the finite convex combinations of the tail sequence $(\varphi_k,\ k\geqslant n)$ in such a way that $(\phi_n',\ n\geqslant 1)$ converges to φ in $\mathbb{D}_{2,2}$. Let $\Lambda(\phi_n')$ be the Jacobian

$$\Lambda(\phi_n') = \det_2(I_H + \nabla^2\phi_n')\exp\left[-\mathcal{L}\phi_n' - \frac{1}{2}|\nabla\phi_n'|_H^2\right].$$

Since the function $A\to -\log\det_2(I_H+A)$ is convex on the space of symmetric Hilbert–Schmidt operators lower bounded by $-I_H$, cf. [1], we have

$$-\log\Lambda(\phi_n') \leqslant \sum_i -t_i\log\Lambda(\varphi_{n_i}), \tag{5}$$

where $\Lambda(\varphi_{n_i})$ is the Gaussian Jacobian of $T_{n_i} = I_W + \nabla\varphi_{n_i}$, and $t_i\geqslant 0$, with $\sum_i t_i = 1$. Recall that $\Lambda(\varphi_n) = 1/L_n\circ T_n,\ T_n = I_W+\nabla\varphi_n,\ (\varphi_n,\ n\geqslant 1)$ converges to φ in $\mathbb{D}_{2,1}$ and $(L_n,\ n\geqslant 1)$ is uniformly integrable. Then a standard argument (cf. [14]) using the Lusin theorem implies that $(1/L_n\circ T_n,\ n\geqslant 1)$ converges to $1/L\circ T$ in probability. Hence the right-hand side of the inequality (5) converges to $\log L\circ T$ and the left-hand side converges to $-\log\Lambda$ in probability. This implies that

$$-\log\Lambda \leqslant \log L\circ T,$$

i.e., $\Lambda\geqslant (L\circ T)^{-1}$ a.s. Since the reverse inequality is already proven in Corollary 1.3, the proof is completed. □

Let us give an immediate corollary:

Corollary 1.5. *We have the following expression for the Wasserstein distance*:

$$\frac{1}{2}d_H^2(\mu,\nu) = -E[f\circ T] - \log c + E\left[\log\det_2(I_H+\nabla^2\varphi)\right] = E[L\log L] + E\left[\log\det_2(I_H+\nabla^2\varphi)\right]. \tag{6}$$

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