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Mathematical Problems in Mechanics

Non-existence of one-dimensional stress problems in solid–solid phase transitions and uniqueness conditions for incompressible phase-transforming materials

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Abstract

We show that for solid–solid phase transitions one-dimensional stress problems do not exist. The lack of uniqueness of solutions in modeling dynamical phase transitions was an unsolved issue. For a slender circular cylinder composed of an incompressible phase-transforming material we establish the proper model equation. From our model equation we establish three relations for three unknowns across the phase boundary, which provide the uniqueness conditions for solutions. Our results seem to resolve the long outstanding issue of nonuniqueness of solutions in modeling dynamical problems of phase-transforming materials. ***To cite this article: H.-H. Dai, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Inexistence des problèmes de contrainte à une dimension dans les transitions de phase solide–solide et conditions d'unicité pour les matériaux incompressibles à transition de phase. Nous montrons que pour les changements de phases solide–solide, il n'existe pas de modèles de contraintes uni-dimensionnel. La non-unicité des solutions des modèles dynamiques de changement de phases était un problème non résolu. Nous obtenons la bonne équation qui modélise un cylindre circulaire mince constitué d'un matériau incompressible à changement de phases. A partir de notre modèle, nous établissons trois relations pour trois inconnues le long de la frontière de phase, qui permettent d'obtenir l'unicité de la solution. Nos résultats semblent résoudre une question restée ouverte pendant longtemps à propos de la non-unicité des solutions des modèles dynamiques de matériaux à changement de phase. ***Pour citer cet article : H.-H. Dai, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

In a recent article [1], an elegent review was given based on the papers by Abeyaratne and Knowles [2–4]. They considered the impact-induced phase transition problem in a semi-infinite slab composed of a phase-transforming material with a given velocity $-V$ at the end. The governing equations (in a Lagrangian description) used by them were pure, one-dimensional dynamical equations

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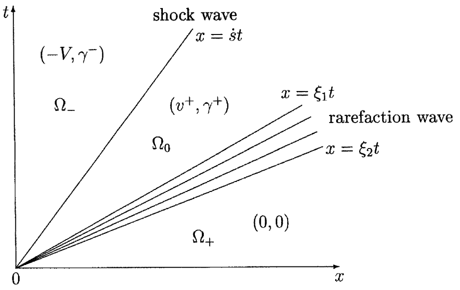

Fig. 1. Impact-induced tensile wave with a similarity form $(v(x/t), \gamma(x/t))$.

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x} = \sigma'(\gamma)\frac{\partial \gamma}{\partial x}, \qquad \frac{\partial \gamma}{\partial t} = \frac{\partial v}{\partial x}, \tag{1}$$

where x and t are respectively the spatial and temporal variables, $\gamma = w_x$ is the axial strain and $v = w_t$ is the velocity (w is the axial displacement), σ is the stress and ρ is the density. For a phase-transforming material, when the given velocity V is within a certain interval, the phase boundary is induced. Abeyaratne et al. [1] gave the solution in the (x, t) plane, which has the structures of one shock wave and a phase boundary. More recently, Knowles [9] considered the case where $\sigma'(\gamma) > 0$ but $\sigma(\gamma)$ has an inflection point. He showed that when the phase boundary is induced, there is also a rarefaction wave; see Fig. 1. The appearence of the rarefaction wave seems to be natural (and probably should also be present in the case where $\sigma'(\gamma)$ changes signs). Across the phase boundary, there are the usual two jump conditions (see [9]):

$$(\gamma^+ - \gamma^-)\dot{s} + (v^+ + V) = 0, \tag{2}$$

$$\sigma(\gamma^+) - \sigma(\gamma^-) + \rho\dot{s}(v^+ + V) = 0, \tag{3}$$

where $\dot{s}$ is the speed of the phase boundary. In the above two equations, v^+ can be related to γ^+ through the stress function $\sigma(\gamma)$, but they only provide two relations for three unknowns γ^+, γ^- and $\dot{s}$. Thus, the solution is not unique and actually there is a one-parameter family of solutions. Abeyaratne and Knowles [2–4] introduced the concept of the driving force $g(t)$ and the kinetic relation $g(t) = \phi(\dot{s})$. Once $\phi(\dot{s})$ is known, then there is an extra relation to determine the solution uniquely. However, as far as the author knows, such a function has never been provided in the literature, except that authors *artificially* used some specific forms to just single out the solutions.

One purpose of this Note is to *establish the uniqueness conditions* through an physical insight into these materials and a formal mathematical derivation.

2. Non-existence of one-dimensional stress problems

Consider the axial equilibrium equation in an axially-symmetrical static problem

$$\frac{\partial S_{zZ}}{\partial Z} + \frac{\partial S_{zR}}{\partial R} + \frac{S_{zR}}{R} = 0, \tag{4}$$

where S_{zZ} and S_{zR} are the components of the first Piolar–Kirchhoff stress tensor, and (r, θ, z) and (R, Θ, Z) are the current and reference cylindrical coordinates, respectively. For a thin bar, one might think that the last two terms in (4) (representing the influence of the radial deformation in the axial direction) are very small and can be neglected. For standard elastic materials, indeed such an approximation is valid (for a neo-Hookean material, they are exponentially small; see [6]). However, for a phase-transforming material, the stress-strain curve typically has a peak-valley combination. In the loading process, as the external stress approaches the peak value σ_p, $\frac{\partial S_{zZ}}{\partial Z} = \frac{\partial S_{zZ}}{\partial \gamma}\gamma_Z$ (γ is the strain) is exactly equal to zero. Then, the terms, $\frac{\partial S_{zR}}{\partial R}$ and $\frac{S_{zR}}{R}$, no matter how small they are, are dominant terms and cannot be neglected! This implies there must be a radial deformation in the process of phase transformation. Thus, to model phase transitions, the influence of the radial deformation should be taken into account. This reveals that *for solid–solid phase transitions, one-dimensional stress problems do not exist.*

3. Model equation and uniqueness conditions

Based on the results given in Section 2, it can be seen that *to model phase transition problems, it is essential to take into account the influence of the radial deformation*. Here, we shall establish the proper model equation for a slender circular cylinder composed of an incompressible phase-transforming material. The idea, similar to that used in [10] for waves in fluids, is to consider which terms should be present and then combine them together.

First, when the phase transformation has not begun, the model equation should be able to yield the correct result for a static uniform state for which the equation has the form $\sigma_Z = 0$. Thus, the term σ_Z should be present.

Secondly, the axial inertial term ρw_{tt} should be present for dynamical problems.

As discussed in Section 2, to model phase transitions, one should take into account the radial deformation. For linear waves, when the lateral movement is present, they are dispersive. Linear dispersive terms are w_{ZZZZ}, w_{ZZtt} and w_{tttt}. Then, combining these terms together, we have the form of the model equation

$$\rho w_{tt} - \sigma_Z + A\mu w_{ZZZZ} + B\rho w_{ZZtt} + C\mu^{-1}\rho^2 w_{tttt} = 0 \tag{5}$$

with three undertermined constants A, B and C, where μ is the shear modulus.

To determine these constants, the idea is to match the dispersion relation of this model equation to the exact dispersion relation based on the three-dimensional field equations up to a certain asymptotic order (cf. [10]). The exact dispersion relation for linear waves in a circular cylinder is the so-called Pochhammer frequency equation (see [5]). For small radius a and the first mode, we expand the Pochhammer frequency equation to $O(a^2)$ and then match it to the dispersion relation of the linearized version of Eq. (5), and as a result we obtain

$$A = \frac{3}{4}a^2, \quad B = -\frac{3}{4}a^2, \quad C = \frac{1}{8}a^2. \tag{6}$$

Thus, the proper model equation for phase transitions in a slender circular cylinder composed of an incompressible phase-transforming material is

$$w_{tt} - \rho^{-1}\sigma_Z + \frac{3}{4}a^2 c_T^2 w_{ZZZZ} - \frac{3}{4}a^2 w_{ZZtt} + \frac{1}{8}a^2 c_T^{-2} w_{tttt} = 0, \tag{7}$$

where c_T is the shear-wave speed. Eq. (7) can also be derived from the three-dimensional field equations together with the traction free boundary conditions in the lateral surface by a consistent asymptotic approach as we have carried out in [7] and [8] for nonlinear waves in slender circular cylinders composed of standard elastic materials. The results based on such an approach for both *compressible* and *incompressible* phase-transforming materials will be reported elsewhere.

For small a, (7) represents a singular perturbation problem and there should be outer and inner regions. Only in the outer regions, at the leading order ($O(1)$) one has Eqs. (1). Thus, the solutions which were constructed based on the pure one-dimensional model in the literature are only valid to within the leading order in the outer regions. In the inner region, the higher-order derivatives come into play, and the full equation should be used.

The solution shown in Fig. 1 is valid in the outer regions. Roughly speaking, one outer region (denoted by R_1) is a region, with the rarefaction wave and the undisturbed region included, some distance away from the phase boundary to the right, and another outer region (denoted by R_2) is some distance away from the phase boundary to the left. The inner region (denoted by I), containing the phase boundary, is between R_1 and R_2. As in the overlapping region of R_1 and I (denoted by D_1) and that of R_2 and I (denoted by D_2) the strain is in a traveling wave state with propagating speed $\dot{s}$, the strain in the whole domain of I should also be in a traveling wave state with the same propagating speed. Thus, we seek the traveling wave solution of (7) and let

$$\gamma(= w_Z) = f(\xi), \quad \xi = Z - \dot{s}t. \tag{8}$$

Integrating the above equation with respect to Z once and then matching to the velocities and strains in the overlapping regions D_1 and D_2, we obtain

$$(\gamma^+ - \gamma^-)\dot{s} + (v^+ + V) = 0. \tag{9}$$

Note that Eq. (9) is precisely the jump condition (2)!

For traveling waves, it is possible to integrate (7) twice, and as a result two integration constants c_1 and c_2 come out. By matching to the velocity and strain values in the overlapping region D_1, one expression for c_1 can be obtained, and by matching those in D_2 another expression for c_1 can be obtained. The two expressions should be equal, which leads to

$$\sigma(\gamma^+) - \sigma(\gamma^-) + \dot{s}(v^+ + V) = 0. \tag{10}$$

This is just the second jump condition (3)! Similarly, by matching to the velocity and strain values in D_1 and D_2, respectively, two expressions for c_2 can be obtained, and setting the two expressions being equal yields that

$$\int_{\gamma^+}^{\gamma^-} \sigma(\gamma)\,\mathrm{d}\gamma = \gamma^-\sigma(\gamma^-) - \gamma^+\sigma(\gamma^+) - \frac{\rho}{2}\dot{s}^2\big(\gamma^{-2} - \gamma^{+2}\big). \tag{11}$$

Eqs. (9), (10) and (11) provide three equations for three unknowns γ^+, γ^- and $\dot{s}$ across the phase boundary, and these are the *uniqueness conditions* for the solution. We point out that these three conditions are derived without using any notion of the kinetic relation.

In the literature, it is stated that the kinetic relation (i.e., the specific form of the driving force in terms of $\dot{s}$) has to be determined from the lattice model and extra experiments, independently from the usual constitutive relation between the stress and strain. Here, we have actually *derived* the uniqueness conditions based on the given stress function $\sigma(\gamma)$ alone. Thus, it is not necessary to conduct extra experiments to determine $\phi(\dot{s})$ or assume artificially any particular forms of $\phi(\dot{s})$ (as adopted by many people in the literature). Our results seem to resolve the long outstanding issue of nonuniqueness of solutions in modeling dynamical problems of phase-transforming materials.

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