Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 338 (2004) 775–780

Comptes Rendus Mathematique

Partial Differential Equations

Solutions concentrating at curves for some singularly perturbed elliptic problems

Andrea Malchiodi

SISSA, via Beirut 2-4, 34014 Trieste, Italy

Received and accepted 16 March 2004

Presented by Haïm Brezis

Abstract

We study positive solutions of the equation $-\varepsilon^2\Delta u + u = u^p$, where $p > 1$ and $\varepsilon > 0$ is small, with Neumann boundary conditions in a three-dimensional domain Ω. We prove the existence of solutions concentrating along some closed curve on $\partial\Omega$. ***To cite this article: A. Malchiodi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Solutions se concentrant sur des courbes pour certains problèmes elliptiques singulières. On étudie les solutions positives de l'équation $-\varepsilon^2\Delta u + u = u^p$, où $p > 1$ et $\varepsilon > 0$ est petit, avec conditions de Neumann sur le bord sur un domaine Ω en dimension 3. On prouve l'existence de solutions qui se concentrent le long de certaines courbes fermées de $\partial\Omega$. ***Pour citer cet article : A. Malchiodi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On considère le problème (P_ε) ci-dessus. Ω est un domaine lisse et borné de $\mathbb{R}^3$, l'exposant p est plus grand que 1 et ε est un réel positif petit. Etant donnée une géodésique simple, fermée, non dégénérée h dans $\partial\Omega$, on prouve l'existence de solutions u_{ε_j} de $(\mathrm{P}_{\varepsilon_j})$ se concentrant sur h le long d'une suite $\varepsilon_j \to 0$. Le profil de u_{ε_j} est donné par la solution radiale de (P_0).

Les fonctions sont obtenues comme points critiques d'une fonctionnelle d'Euler I_ε définie sur $H^1(\Omega)$. Leur energie est grande quand on la compare à celle des niveaux obtenus par mountain-pass level, et leur indice de Morse tend vers l'infini lorsque $\varepsilon_j \to 0$. La preuve est basée sur un argument d'inversion locale, combiné avec une étude fine du spectre de I''_ε au voisinage de solutions approchées $\tilde{u}_\varepsilon$ de (P_ε) qui possèdent le profile requis. Puisque l'indice de Morse de $\tilde{u}_\varepsilon$ change avec ε, $I''_\varepsilon(\tilde{u}_\varepsilon)$ peut avoir des valeurs propres nulles dépendant de ε. Notre analyse nous autorise à selectionner une suite ε_j le long de laquelle l'opérateur linéarisé est inversible.

Cette Note concerne les résultats obtenus dans [11]. Nous donnons ici le théorème principal et les idées de la preuve.

E-mail address: malchiod@sissa.it (A. Malchiodi).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.03.023

1. Introduction

Given $\Omega \subseteq \mathbb{R}^n$, $p > 1$ and a small positive parameter ε, we consider the following equation

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{P_ε}$$

Problem (P_ε) arises in the study of a class of reaction–diffusion systems which model some experiments in chemistry or biology, see [16]. For example, when two chemical substances interact and have very different diffusivities, at equilibrium one of them can be nearly constant through Ω. The distribution of the second substance, instead, may be very inhomogeneous and is described by an equation like (P_ε). Typically, as ε tends to zero, one expects solutions to become sharply concentrated near some regions of Ω.

Much work has been devoted to (P_ε) in order to understand where concentration occurs and what the profile of solutions looks like. In the case $p < \frac{n+2}{n-2}$, due to several contributions (starting from the seminal works [10,17,18]) already the structure of solutions concentrated at points, called *spike-layers*, has been shown to be very rich. For example, in [7], Gui and Wei construct solutions with l_1 peaks at the boundary of Ω and l_2 peaks in the interior, where l_1 and l_2 are arbitrary integer numbers. Naively, peaking at $\partial\Omega$ occurs at critical points of the mean curvature while peaking in the interior occurs at critical points of the distance from the boundary.

Similar results hold for the above singularly perturbed problem with Dirichlet boundary conditions, or the Nonlinear Schrödinger equation in the semiclassical limit, see, e.g., [6,9] and references therein.

Only recently existence of solutions concentrating at different sets has been proved. It is indeed conjectured, see [16], that for any integer $k \in \{1, \dots, n-1\}$ problem (P_ε) generically admits solutions concentrating at k-dimensional sets. In [12,13] the authors have shown that, given any smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, and any $p > 1$, problem (P_ε) admits solutions concentrating at the whole boundary along a suitable sequence $\varepsilon_j \to 0$.

We describe here a first result concerning concentration at a curve on the boundary of a three-dimensional domain, namely at a set of codimension 2.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain, and let $p > 1$. Let $h : S^1 \to \partial\Omega$ be a closed non-degenerate geodesic without self-intersections. Then there exists a sequence $\varepsilon_j \to 0$ and a sequence of solutions u_{ε_j} of $(\mathrm{P}_{\varepsilon_j})$ which concentrates at h. The profile of u_{ε_j} is given by the* (*unique*) *radial solution of the problem*

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^2_+, \\ u > 0 & u \in H^1(\mathbb{R}^2_+). \end{cases} \tag{P_0}$$

For reasons of brevity, we do not specify rigorously what the above expression *concentrates at h* means. Some more details are given in the next sections.

Remark 1. (a) We point out that in the theorem no upper bound on p is assumed, hence the exponent could also be supercritical. This is due to the fact that the limit equation (P_0), which gives the profile of u_{ε_j}, is in $\mathbb{R}^2$, and for that there is no restriction for the existence of a solution. In the case of solutions concentrating at points, instead, the subcriticality of p is a necessary condition, see [3].

(b) The choice of a *specific* sequence ε_j is fundamental for our approach, and is not used to recover compactness. It is required to obtain the invertibility of the linearized equation at some approximate solutions $\tilde{u}_{\varepsilon_j}$. This invertibility is false in general, and we do not expect that generically concentration at a k-dimensional of solutions with the same profile can occur for all the values of ε. Analogous phenomena occur in the construction of constant mean curvature tubes shrinking at curves or manifolds, see [15].

(c) By construction, the Morse index of u_{ε_j} tends to infinity as ε_j tends to zero. This should be compared to a result in [4], where it is proven that a family of solutions of (P_ε) with uniform bounds on the Morse index must concentrate at a finite number of points when ε tends to zero.

(d) Under some symmetry assumptions, solutions concentrating at manifolds may have bounded Morse index in the space of invariant functions. In this case, as for the study of concentration at points, it is possible to use minimax methods and finite-dimensional reductions in order to prove existence, see for example [1,2,5,14,19].

Theorem 1.1 is proved with a local inversion argument. The first step, described in Section 2, consists in finding an approximate solution $\tilde{u}_\varepsilon$. Then in Section 3 an accurate analysis of the linearized equation at $\tilde{u}_\varepsilon$ is performed, focusing on its spectral properties. Finally, Section 4 is devoted to the proof of Theorem 1.1, via the Contraction Mapping Theorem.

The present Note concerns the result obtained in [11], where the procedure is carried out in full detail.

2. Approximate solutions

When $p \leqslant 5$ (the subcritical or critical case, recall that $\Omega \subseteq \mathbb{R}^3$), solutions of (P_ε) can be found as critical points of the functional $I_\varepsilon : H^1(\Omega) \to \mathbb{R}$ defined by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[\varepsilon^2|\nabla u|^2 + u^2\right] - \frac{1}{p+1}\int_\Omega |u|^{p+1}. \tag{1}$$

In order to find approximate solutions $\tilde{u}_\varepsilon$, namely functions for which $\|I'_\varepsilon(\tilde{u}_\varepsilon)\|$ is small, we introduce some local coordinates on $\partial\Omega$ near the geodesic h. Without loss of generality we can assume that the length of h is 2π and that $t \mapsto h(t)$ is a parametrization by unit length defined on S^1. Since h is simple, the map $\Phi : S^1 \times [-\delta, \delta]$ (where δ is a small number) defined below gives a system of coordinates for $\partial\Omega$ in a neighborhood of h. For any $t \in S^1$, let $e(t) \in T_{h(t)}\partial\Omega$ be a unit vector tangent to $\partial\Omega$ and satisfying $\dot{h}(t) \wedge e(t) = \nu_{h(t)}$, where $\nu_{h(t)}$ is the inner unit normal to $\partial\Omega$ at $h(t)$.

Letting exp denote the exponential map on $\partial\Omega$, we define $\Phi : S^1 \times [-\delta, \delta] \to \partial\Omega$ by

$$\Phi(x_1, x_2) = \exp_{h(t)}\bigl(x_2 e(x_1)\bigr), \quad (x_1, x_2) \in S^1 \times [-\delta, \delta]. \tag{2}$$

If $\bar{g}$ is the metric of $\partial\Omega$ induced by $\mathbb{R}^3$, we let $\bar{g}_{ij}$ be the coefficients of $\bar{g}$ in the above coordinates. From the non-degeneracy of h one can obtain the following result.

Lemma 2.1. *Let (x_1, x_2) be the coordinates given by* (2). *Then the bilinear form*

$$(u, v) \mapsto \int_{S^1} \dot{u}\dot{v} + \frac{1}{2}\int_{S^1} \partial^2_{x_2}\bar{g}_{11}(x_1, 0)uv$$

is non-degenerate on $H^1(S^1) \times H^1(S^1)$.

Using the above map Φ, we also define some parametrization of the *interior* of Ω in a neighborhood of h. We introduce the map $\tilde{\Phi}$ given by

$$\tilde{\Phi}(x_1, x_2, x_3) = \Phi(x_1, x_2) + \nu(x_1, x_2)x_3, \quad x = (x_1, x') \in S^1 \times [-\delta, \delta] \times [0, \delta]. \tag{3}$$

Here $\nu(x_1, x_2)$ denotes the inner unit normal to $\partial\Omega$ at $\Phi(x_1, x_2)$.

Using these coordinates, the approximate solutions $\tilde{u}_\varepsilon$ will have the form $\tilde{u}_\varepsilon(x) \simeq w_0(x'/\varepsilon)$, $x' = (x_2, x_3)$, where w_0 is the unique radial solution of (P_0). More precisely, given $k \in \mathbb{N}$, we take

$$\begin{aligned}\tilde{u}_\varepsilon := u_{k,\varepsilon} = {} & w_0\bigl(x_2/\varepsilon + f_0(x_1) + \cdots + \varepsilon^{k-2} f_{k-2}(x_1), x_3/\varepsilon\bigr) \\ & + \varepsilon w_1(x_1, x'/\varepsilon) + \cdots + \varepsilon^k w_k(x_1, x'/\varepsilon) + \mathrm{o}\bigl(\varepsilon^k\bigr),\end{aligned} \tag{4}$$

where $f_1,\dots,f_{k-2}$ and $w_1,\dots,w_k$ are suitable functions to be determined. This is done expanding formally the equation $-\varepsilon^2\Delta\tilde u_\varepsilon + \tilde u_\varepsilon = \tilde u_\varepsilon^p$, with the choice of (4), in powers of ε. In this way one can find recursively the functions $\{f_i\}$ and $\{w_i\}$. The term of order ε^i in the expansion will indeed determine w_i and f_{i-2}. In order to give an idea of the method, we recall the following well-known result.

Lemma 2.2. *Let L be linearization of* (P_0) *at the ground-state solution w_0, namely $Lv = -\Delta v + v - pw_0^{p-1}v$ in $\mathbb{R}^2_+$; $\frac{\partial v}{\partial \nu} = 0$ on $\partial\mathbb{R}^2_+$. Then L is self-adjoint and its kernel in $H^1(\mathbb{R}^2_+)$ is generated by $\partial w_0/\partial x_2$.*

The expansion of the ith term in the equation, after some scaling, will be of the form $Lw_i = F(x_1, f_{i-2}(x_1), f''_{i-2}(x_1))$ in $S^1 \times \mathbb{R}^2_+$; $\frac{\partial w_i}{\partial \nu} = 0$ on $S^1 \times \partial\mathbb{R}^2_+$. Here, naively, $\mathbb{R}^2_+$ replaces the expanding rectangle $[-\delta/\varepsilon, \delta/\varepsilon] \times [0, \delta/\varepsilon]$ in the x' variable. By Lemma 2.2, a necessary condition for the existence is the orthogonality of the right-hand side to $\partial w_0/\partial x_2$ for every $x_1 \in S^1$. This is guaranteed if f_{k-2} satisfies some ODE of the second order, whose solvability is a consequence of Lemma 2.1. All these arguments can be made rigorous to give the following result.

Proposition 2.3. *Consider the Euler functional I_ε defined in* (1). *Then for any $k \in \mathbb{N}$ there exist functions $\tilde u_\varepsilon : \Omega \to \mathbb{R}$ with the following properties*

$$\left\| I'_\varepsilon(\tilde u_\varepsilon)\right\|_{H^1(\Omega)} \leqslant C_k \varepsilon^{k+1/2}; \qquad \tilde u_\varepsilon \geqslant 0 \quad \text{in } \Omega; \qquad \frac{\partial \tilde u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where C_k depends only on Ω, p and k.

3. Study of the linearized operator

We now focus on $I''_\varepsilon(\tilde u_\varepsilon)$. Our aim is to prove that along some sequence $\varepsilon_j \to 0$ this operator is invertible, having some quantitative estimates on the norm of the inverse. This is the content of Proposition 3.3.

Our method relies on viewing the eigenvalues as functions of ε. In order to understand this dependence the following result, due to Kato, see [8], is very useful.

Proposition 3.1. *Let $T(\chi)$ denote a differentiable family of operators from an Hilbert space H into itself, where χ lies in a neighborhood of* 0. *Let $T(0)$ be a self-adjoint operator of the form* Identity-compact, *and let $\zeta(0) = \zeta_0 \neq 1$ be an eigenvalue of $T(0)$. Then the eigenvalue $\zeta(\chi)$ is differentiable at* 0 *with respect to χ. The* (*possibly multivalued*) *derivative of ζ is given by $\frac{\partial \zeta}{\partial \chi} = \{$eigenvalues of $P_{\zeta_0} \circ \frac{\partial T}{\partial \chi}(0) \circ P_{\zeta_0}\}$, where $P_{\zeta_0} : H \to H_{\zeta_0}$ denotes the projection onto the ζ_0-eigenspace H_{ζ_0} of $T(0)$.*

Note that, in order to apply Proposition 3.1, some information on the eigenspace H_{ζ_0} is required. To have that in our case, we construct first a family of approximate eigenfunctions $\{\Psi_j^\varepsilon\}_j$, $\{\tilde\Psi_k^\varepsilon\}_k$ of $I''_\varepsilon(\tilde u_\varepsilon)$. Consider the eigenvalue problems

$$-\phi''_j = \omega_j \phi_j \quad \text{in } S^1; \qquad -\psi''_k + a(t)\psi_k = \lambda_k \psi_k \quad \text{in } S^1, \tag{6}$$

with periodic boundary conditions, where $a(t) = \frac{1}{2}\partial^2_{x_2}\bar g_{11}(t,0)$ (see the previous section). We note that ϕ_j are just elementary trigonometric functions, and the numbers ω_j can be written explicitly. Also, from Lemma 2.1 it follows that all the numbers λ_k are different from zero.

If w_0 is as above and $\alpha > 0$, we let μ_α denote the first eigenvalue (which is simple) of the problem

$$-\Delta v + (1+\alpha)v - pw_0^{p-1} v = \mu_\alpha(-\Delta v + v) \quad \text{in } \mathbb{R}^2_+; \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathbb{R}^2_+, \tag{7}$$

with corresponding eigenfunction v_α. Then one can show that, in the above coordinates, functions of the form $\Psi_j^\varepsilon(x) = \phi_j(x_1) v_{\varepsilon^2\omega_j}(x'/\varepsilon) + \mathrm{O}(\varepsilon)$; $\tilde\Psi_k^\varepsilon = \psi_k(x_1)\partial_2 w_0(x'/\varepsilon) + \mathrm{O}(\varepsilon)$ satisfy the approximate eigenvalue equations

$$I''_\varepsilon\big(\Psi_j^\varepsilon\big) \simeq \mu_{j,\varepsilon}\Psi_j^\varepsilon; \qquad I''_\varepsilon\big(\tilde\Psi_k^\varepsilon\big) = \varepsilon^2\lambda_j \tilde\Psi_k^\varepsilon + \mathrm{o}\big(\varepsilon^2\big). \tag{8}$$

Naively, the functions Ψ_j^ε and $\tilde{\Psi}_k^\varepsilon$ can be thought as longitudinal and transversal *modes of vibration* of $\tilde{u}_\varepsilon$. Since this is concentrated near the geodesic h, in the second equation of (8) one obtains the eigenvalues of the Jacobi operator (the second variation of the length).

The numbers $\mu_{j,\varepsilon}$ in the last formula behave *qualitatively* in the following way

$$\mu_{j,\varepsilon} \sim -(p-1) + \varepsilon^2 j^2 + \mathrm{O}(\varepsilon). \tag{9}$$

Since we want to get a control of order ε^2 on the eigenvalues of $I_\varepsilon''(\tilde{u}_\varepsilon)$, the last estimate is not accurate enough, and that's why we need to use Proposition 3.1. In order to apply it, we need to recover information on the true eigenfunctions of $I_\varepsilon''(\tilde{u}_\varepsilon)$. We consider the three subspaces of $H^1(\Omega)$ $H_1 = \mathrm{span}\{\Psi_j^\varepsilon\colon\ j = 1, \ldots, \widehat{C}_0\}$; $H_2 = \mathrm{span}\{\tilde{\Psi}_k^\varepsilon\colon\ k = 1, \ldots, \varepsilon^{-\xi}\}$; $H_3 = (H_1 \oplus H_2)^\perp$, where $\widehat{C}_0$ and ξ are numbers to be chosen appropriately. One can prove that, for good choices of $\widehat{C}_0$ and ξ and for ε small, every $u \in H^1(\Omega)$ decomposes uniquely as $u = u_1 + u_2 + u_3$, where $u_i \in H_i$ for all i. We apply this decomposition to the eigenvectors of $I_\varepsilon''(\tilde{u}_\varepsilon)$ corresponding to eigenvalues which are close to zero.

Lemma 3.2. *There exists $\delta > 0$ sufficiently small with the following property. Let v be an eigenfunction of $I_\varepsilon''(\tilde{u}_\varepsilon)$ with eigenvalue $\lambda \in [-\delta\varepsilon^2, \delta\varepsilon^2]$. Then, writing $v = v_1 + v_2 + v_3$, one has that $\|v_2\| = \mathrm{o}(1)$ and $\|v_3\| = \mathrm{o}(1)$. Furthermore, there holds $\frac{\partial\lambda}{\partial\varepsilon} = \frac{1}{\varepsilon}(C_0 + \mathrm{o}(1))$ $(\mathrm{o}(1) \to 0$ as $\varepsilon \to 0)$, where we are considering λ as a function of ε and where C_0 is some constant depending on p.*

The first assertion of this lemma is proved by testing the eigenvalue equation $I_\varepsilon''(\tilde{u}_\varepsilon)v = \lambda v$ on the components v_1, v_2 and v_3. Once this is done, one also finds that the Fourier modes of v in H_1 are mainly localized near $j \sim \frac{\overline{C}_0}{\varepsilon}$, where $\overline{C}_0$ is another constant depending on p. Basically, those are the modes for which the approximate eigenvalues $\mu_{\varepsilon,j}$, see (9), are close to zero. Hence, having a precise characterization of the eigenfunction v, we can apply Proposition 3.1 and get the second statement.

Lemma 3.2 is applied in the following way. From the asymptotics of $\mu_{j,\varepsilon}$ in (9) one finds gaps of with ε^2 in the spectrum simply by counting the number of eigenvalues in a given interval. Then, varying ε suitably, these gaps can be brought near zero keeping their size nearly constant.

Proposition 3.3. *Let $\tilde{u}_\varepsilon$ be as in Proposition 2.3. Then for a suitable sequence $\varepsilon_j \to 0$, the operator $I_{\varepsilon_j}''(\tilde{u}_{\varepsilon_j}) : H^1(\Omega) \to H^1(\Omega)$ is invertible and the inverse operator satisfies $\|I_{\varepsilon_j}''(\tilde{u}_{\varepsilon_j})^{-1}\| \leqslant \frac{C}{\varepsilon_j^2}$ for all $j \in \mathbb{N}$.*

Remark 2. From our arguments, one can show that the set of values ε for which $I_\varepsilon''(\tilde{u}_\varepsilon)$ is invertible (and for which our method produces solutions of (P_ε)) it not only a sequence, but it has density converging to 1 in smaller and smaller right neighborhoods of the origin.

4. Proof of Theorem 1.1

Let ε_j be as in Proposition 3.3. For brevity, in the rest of the proof, we simply write ε instead of ε_j. Recall that so far we have assumed that $p \leqslant 5$. Now we just apply the contraction mapping theorem, looking for a solution u_ε of the form $u_\varepsilon = \tilde{u}_\varepsilon + w$, $w \in H^1(\Omega)$. Since $I_\varepsilon''(\tilde{u}_\varepsilon)$ is invertible (along the sequence ε_j), we can write

$$I_\varepsilon''(\tilde{u}_\varepsilon + w) = 0 \quad \Leftrightarrow \quad w = -\big(I_\varepsilon''(\tilde{u}_\varepsilon)\big)^{-1}\big[I_\varepsilon'(\tilde{u}_\varepsilon) + G(w)\big], \tag{10}$$

where $G(w) = I_\varepsilon'(\tilde{u}_\varepsilon + w) - I_\varepsilon'(\tilde{u}_\varepsilon) - I_\varepsilon''(\tilde{u}_\varepsilon)[w]$. Note that $G(w)[v] = -\int_\Omega [(\tilde{u}_\varepsilon + w)^p - \tilde{u}_\varepsilon^p - p\tilde{u}_\varepsilon^{p-1}w]v$, $v \in H^1(\Omega)$, hence $G(w)$ is basically superlinear in w. Let us define the operator $F_\varepsilon : H^1(\Omega) \to H^1(\Omega)$ as $F_\varepsilon(w) = -(I_\varepsilon''(\tilde{u}_\varepsilon))^{-1}[I_\varepsilon'(\tilde{u}_\varepsilon) + G(w)]$, $w \in H^1(\Omega)$. We are going to prove that F_ε is a contraction in a suitable closed set of $H^1(\Omega)$. From Proposition 2.3 and some elementary inequalities one finds

$$\|F_\varepsilon(w)\| \leqslant \begin{cases} C\varepsilon^{-2}\big(\varepsilon^{k+1/2} + \|w\|^p\big) & \text{for } p \leqslant 2, \\ C\varepsilon^{-2}\big(\varepsilon^{k+2/2} + \|w\|^2\big) & \text{for } p > 2, \end{cases} \quad \|w\| \leqslant 1; \tag{11}$$

$$\|F_\varepsilon(w_1) - F_\varepsilon(w_2)\| \leqslant \begin{cases} C\varepsilon^{-2}\big(\|w_1\|^{p-1} + \|w_2\|^{p-1}\big)\|w_1 - w_2\|, & p \leqslant 2, \\ C\varepsilon^{-2}\big(\|w_1\| + \|w_2\|\big)\|w_1 - w_2\|, & p > 2, \end{cases} \quad \|w_1\|, \|w_2\| \leqslant 1. \tag{12}$$

Now we choose integers d and k such that

$$d > \begin{cases} \dfrac{2}{p-1} & \text{for } p \leqslant 2, \\ 2 & \text{for } p > 2, \end{cases} \qquad k > d + \frac{3}{2}, \tag{13}$$

and we set $\mathcal{B} = \{w \in H^1(\Omega)\colon \|w\| \leqslant \varepsilon^d\}$. From (11) and (12) we find that F_ε is a contraction in $\mathcal{B}$ for ε small, and the existence of a solution u_ε follows. This function, by construction, will have the required asymptotics. The positivity of u_ε can be proved in a standard way, testing the equation on $(u_\varepsilon)^-$ and using the Sobolev embeddings. This concludes the proof in the case $p \leqslant 5$. For larger p see the remark below.

Remark 3. The proof in the supercritical case goes along the same way, but with some modification. One can consider some truncated functional, whose nonlinearity is still subcritical for $|u|$ large. Then, the contraction argument is performed in a set of functions with small H^1 norm and small L^∞ norm. Using elliptic regularity estimates, then one obtains an upper bound in L^∞ of the solutions, which is independent on the truncation. Therefore, critical points of the modified Euler functional will be true solutions of (P_ε) even for $p > 5$.

Acknowledgements

This note has been written when the author was visiting the university of Paris VI, under the European Grant ERB FMRX CT98 0201. He wishes to thank the institution for the kind hospitality.

References

[1] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Commun. Math. Phys. 235 (2003) 427–466.
[2] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part II, Indiana Univ. Math. J., in press.
[3] S. Cingolani, A. Pistoia, Nonexistence of single blow-up solutions for a nonlinear Schrödinger equation involving critical Sobolev exponent, Z. Angew. Math. Phys., in press.
[4] E.N. Dancer, Stable and finite Morse index solutions on $\mathbb{R}^n$ or on bounded domains with small diffusion, Preprint.
[5] T. D'Aprile, Differential Integral Equations 16 (3) (2003) 349–384.
[6] M. Del Pino, P. Felmer, J. Funct. Anal. 149 (1997) 245–265.
[7] C. Gui, J. Wei, Canad. J. Math. 52 (3) (2000) 522–538.
[8] T. Kato, Perturbation Theory for Linear Operators, second ed., in: Grundlehren Math. Wiss., vol. 132, Springer-Verlag, Berlin, 1976.
[9] Y.Y. Li, L. Nirenberg, Comm. Pure Appl. Math. 51 (1998) 1445–1490.
[10] C.-S. Lin, W.-M. Ni, I. Takagi, J. Differential Equations 72 (1988) 1–27.
[11] A. Malchiodi, Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains, Preprint.
[12] A. Malchiodi, M. Montenegro, Comm. Pure Appl. Math. 15 (2002) 1507–1568.
[13] A. Malchiodi, M. Montenegro, Multidimensional boundary-layers for a singularly perturbed Neumann problem, Duke Univ. Math. J., in press.
[14] A. Malchiodi, W.-M. Ni, J. Wei, Multiple clustered layer solutions for semilinear Neumann problems on a ball, Preprint.
[15] R. Mazzeo, F. Pacard, Foliations by constant mean curvature tubes, Preprint.
[16] W.-M. Ni, Notices Amer. Math. Soc. 45 (1) (1998) 9–18.
[17] W.-M. Ni, I. Takagi, Comm. Pure Appl. Math. 41 (1991) 819–851.
[18] W.-M. Ni, I. Takagi, Duke Math. J. 70 (1993) 247–281.
[19] J. Shi, Trans. Amer. Math. Soc. 354 (8) (2002) 3117–3154.