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Number Theory

On the special values of automorphic L-functions

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Abstract

Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with non-vanishing cohomology and denote by $L(\pi, s)$ its L-function. Under a certain local non-vanishing assumption, we prove the rationality of the values of $L(\pi \otimes \chi, 0)$ for characters χ, which are critical for π. Note that conjecturally any motivic L-function should coincide with an automorphic L-function on GL_n; hence, our result corresponds to a conjecture of Deligne for motivic L-functions. ***To cite this article: J. Mahnkopf, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur les valeurs spéciales des fonctions L automorphes. Soit π une représentation cuspidale de $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ dont la cohomologie d'algèbre de Lie relative ne s'annule pas et soit $L(\pi, s)$ sa fonction L automorphe. Sous l'hypothèse qu'une certaine intégrale locale ne s'annule pas nous démontrons la rationalité des valeurs $L(\pi \otimes \chi, 0)$ pour les charactères χ, qui sont critiques pour π. Notons que conjecturalement chaque fonction L motivique est égale a une fonction L automorphe attachée à GL_n, donc, notre résultat correspond à une conjecture de Deligne concernant les fonctions L motiviques. ***Pour citer cet article : J. Mahnkopf, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Statement of results

We shall use the following notations. $\mathbb{A} = \prod_v \mathbb{Q}_v$ is the ring of adeles of $\mathbb{Q}$. B_n resp. T_n resp. Z_n denotes the subgroup of GL_n consisting of upper triangular matrices resp. diagonal matrices resp. the center of GL_n. Throughout we assume $n \geqslant 3$. We write $X^+(T_n)$ to denote the set of dominant (with respect to B_n) algebraic characters of T_n and (ρ_μ, M_μ) is the irreducible representation of GL_n of highest weight $\mu \in X^+(T_n)$. We set $\mu^\vee = -w_n\mu$, where w_n is the longest element in the Weyl group W_{GL_n} of GL_n; $\mu^\vee$ then is the highest weight of the contragredient representation $(\rho^\vee, M_\mu^\vee)$. For any field F containing $\mathbb{Q}$ we set $M_{\mu,F} = M_\mu \otimes F$. Moreover we denote by $\mathrm{Coh}(\mathrm{GL}_n, \mu)$ the set of all cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A})$ such that

$$H^\bullet\big(\mathfrak{gl}_n, \mathrm{SO}_n(\mathbb{R})Z_n(\mathbb{R})^0, \pi_\infty \otimes M_\mu\big) \neq 0.$$

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Since cuspidal representations are quasi unitary, $\mathrm{Coh}(\mathrm{GL}_n, \mu)$ is non-empty only if the weight $\mu = (\mu_i)$ satisfies the following purity condition: there is an integer $\mathrm{wt}(\mu) \in \mathbb{Z}$ such that

$$\mu_i + \mu_{n+1-i} = \mathrm{wt}(\mu), \quad 1 \leqslant i \leqslant n.$$

In case n odd this implies $\mathrm{wt}(\mu) \in 2\mathbb{Z}$. We note that a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ with non-vanishing relative Lie-Algebra cohomology is defined over a finite extension $E_\pi/\mathbb{Q}$ (cf. [2], Théorème 3.13). Furthermore, these representations are *algebraic* in the sense of [2], Définition 1.8.

Let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ and let $L(\pi, s) = \prod_v L(\pi_v, s)$ be the automorphic L-function attached to it. Any complex character χ_∞ of $\mathbb{R}^*$ is of the form $\chi_\infty = \varepsilon_\infty |\cdot|_\infty^k$, where ε_∞ is of order $\leqslant 2$ and $k \in \mathbb{C}$. We say that χ_∞ is critical for π_∞ if • $k \in 1/2 + \mathbb{Z}$ if n is even and $k \in \mathbb{Z}$ if n is odd • $L(\pi_\infty \otimes \chi_\infty, 0)$ and $L(\pi_\infty^\vee \otimes \chi_\infty^{-1}, 1)$ are regular values (note that under the conjectural correspondence between motives M and algebraic automorphic representations π on GL_n we have $L(M, s) = L(\pi, s + (n-1)/2)$; cf. [2], 4.5, 4.16). We say that $\chi : \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}^*$ is critical for π if χ_∞ is critical for π_∞. We denote the set of critical characters χ_∞ resp. χ by $\mathrm{Crit}(\pi_\infty)$ resp. $\mathrm{Crit}(\pi)$. Using the classification of (generic) unitary $(\mathfrak{gl}_n, \mathrm{SO}_n)$-modules with non-vanishing cohomology we find:

Proposition 1.1. *Let $\mu = (\mu_i) \in X^+(T_n)$ and let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu^\vee)$. Then, $\chi_\infty = \varepsilon_\infty |\cdot|_\infty^k$ is critical for π precisely if*

- $-\mu_{n/2} + 1/2 \leqslant k \leqslant \mu_{n/2} + 1/2 - \mathrm{wt}(\mu)$ *if n is even*;
- $-\mu_{(n-1)/2} \leqslant k \leqslant \mu_{(n-1)/2} + 1 - \mathrm{wt}(\mu)$ *if n is odd.*

In case that n is odd χ in addition has to satisfy a parity condition: denote by ω_π the central character of π and by $\omega_\pi|_{\pi_0}$ its restriction to $\{\pm\mathbf{1}\} \subset Z_n(\mathbb{R})$. Put $l = (\mathrm{wt}(\mu) - n + 1)/2 + k$; then χ_∞ has to satisfy $\varepsilon_\infty = \omega_\pi|_{\pi_0} \mathrm{sgn}^l$ if $k > (1 + \mathrm{wt}(\mu))/2$ and $\varepsilon_\infty = \omega_\pi|_{\pi_0} \mathrm{sgn}^{l+1}$ if $k \leqslant (1 + \mathrm{wt}(\mu))/2$.

Obviously, χ is critical for π precisely if $\chi^{-1}|\cdot|$ is critical for the contragredient representation $\pi^\vee$. In view of the functional equation relating $L(\pi \otimes \chi, 0)$ to $L(\pi^\vee \otimes \chi^{-1}|\cdot|, 0)$ it is therefore sufficient to consider points $\chi \in \mathrm{Crit}(\pi)$ with component at infinity $\chi_\infty = \varepsilon_\infty |\cdot|_\infty^k$ satisfying $k \leqslant (1 - \mathrm{wt}(\mu))/2$. We denote the set of critical characters satisfying this condition by $\mathrm{Crit}(\pi_\infty)^\leqslant$ and $\mathrm{Crit}(\pi)^\leqslant$.

We define a collection of complex numbers $\Omega(\pi, \chi_\infty) \in \mathbb{C}^*/\widehat{E}_\pi^*$, where $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ and $\chi_\infty \in \mathrm{Crit}(\pi_\infty)^\leqslant$ such that for any finite extension $E/\widehat{E}_\pi$ the tuple

$$\{\Omega(\pi^\sigma, \chi_\infty)\}_{\sigma \in \mathrm{Hom}(E, \mathbb{C})} \in (E \otimes \mathbb{C})^*/\widehat{E}_\pi^*$$

is well defined. Here, $\widehat{E}_\pi/E_\pi$ is a certain finite extension and $\widehat{E}_\pi$ is embedded as $\alpha \mapsto \{\sigma(\alpha)\}_{\sigma \in \mathrm{Hom}(E, \mathbb{C})}$. We set $\chi^1 = \chi|\cdot|^{-k}$, where $\chi_\infty = \varepsilon_\infty |\cdot|_\infty^k$ and we define $\chi^\sigma = \sigma(\chi|\cdot|^{-k})|\cdot|^k$, $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Theorem 1.1. *Assume that $\mu \in X^+(T_n)$ is regular and let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu^\vee)$.*

(1) *For all but finitely many $\chi \in \mathrm{Crit}(\pi)^\leqslant$ we have*

$$L(\pi \otimes \chi, 0) = \Omega(\pi, \chi_\infty) \mod \widehat{E}_\pi(\chi^1).$$

Moreover, denote by $G(\chi)$ the Gauss sum attached to χ. For all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$\left(G(\chi)^{[n/2]} \frac{L(\pi \otimes \chi, 0)}{\Omega(\pi, \chi_\infty)}\right)^\sigma = G(\chi^\sigma)^{[n/2]} \frac{L(\pi^\sigma \otimes \chi^\sigma, 0)}{\Omega(\pi^\sigma, \chi_\infty)}.$$

(2) *Let $\chi'_\infty = \varepsilon'_\infty |\cdot|_\infty^{k'} \in \mathrm{Crit}(\pi)^\leqslant$. Then, the ratio $\Omega(\pi, \chi_\infty)/\Omega(\pi, \chi'_\infty)$ only depends on χ_∞, χ'_∞ and μ; in particular, it does not depend on π (obviously, in case n odd it is also independent of ε_∞ and ε'_∞).*

Remark 1. (a) The theorem is valid only under a certain non-vanishing assumption (cf. below). This assumption is known to hold in case $n=3$ and is analogous to the assumption made in [1], p. 28.

(b) In cases $n=1,2$ (which we have excluded) Theorem 1.1 has previously been known to hold (cf. [4]).

The proof of Theorem 1.1 uses an induction over the rank of GL_n. To be more precise, let $\mu\in X^+(T_n)$. We select a weight $\lambda\in X^+(T_{n-1})$ such that • $\lambda\preceq\mu$, i.e., $\mu_1\geqslant\lambda_1\geqslant\mu_2\geqslant\cdots\geqslant\lambda_{n-1}\geqslant\mu_n$ • $\lambda_{n/2}=-k+1/2$ if n is even and $\lambda_{(n+1)/2}=-k$ if n is odd. Proposition 1.1 implies that such a choice of λ is possible precisely if k is a critical point for π (i.e., $\varepsilon_\infty|\cdot|_\infty^k\in\mathrm{Crit}(\pi)$ for some ε_∞). We denote by $P\leqslant\mathrm{GL}_{n-1}$ the parabolic subgroup of type $(n-2,1)$ containing B_{n-1} and by $W^P\subset W_{\mathrm{GL}_{n-1}}$ the relative Weyl group, i.e., W^P is a system of representatives for $W_{M_P}\backslash W_{\mathrm{GL}_{n-1}}$. We set

$$\widehat{w}=\begin{pmatrix}1 & 2 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \cdots & n-1\\ 1 & 2 & \cdots & \left[\frac{n}{2}\right]-1 & n-1 & \left[\frac{n}{2}\right] & \cdots & n-2\end{pmatrix}\in W^P.$$

We define the weight $\mu'=\widehat{w}(\lambda+\rho_{n-1})-\rho_{n-1}|_{T_{n-2}}\in X^+(T_{n-2})$, where ρ_{n-1} is half the sum of the positive roots of GL_{n-1} determined by B_{n-1} and we embed $T_{n-2}\hookrightarrow T_{n-1}$ via $t\mapsto\mathrm{diag}(t,1)$. Using the $\mathbb{Q}$-structure on the cohomology of locally symetric spaces we define a collection of complex numbers $\Omega(\pi,\pi',\varepsilon_\infty)\in\mathbb{C}^*/(E_\pi E_{\pi'})^*$, where $\pi\in\mathrm{Coh}(\mathrm{GL}_n,\mu)$, $\pi'\in\mathrm{Coh}(\mathrm{GL}_{n-2},\mu')$ and ε_∞ is a character of order $\leqslant 2$ such that for any finite extension $E/E_\pi E'_\pi$ the tuple $\{\Omega(\pi^\sigma,\pi'^\sigma,\varepsilon_\infty)\}_{\sigma\in\mathrm{Hom}(E,\mathbb{C})}\in(E\otimes\mathbb{C})^*/(E_\pi E_{\pi'})^*$ is well defined.

Theorem 1.2. *Assume that $\mu\in X^+(T_n)$ is regular and let $\pi\in\mathrm{Coh}(\mathrm{GL}_n,\mu^\vee)$ and $\pi'\in\mathrm{Coh}(\mathrm{GL}_{n-2},\mu')$ (if n is odd π' in addition has to satisfy a parity condition).*

(1) *For all $\chi\in\mathrm{Crit}(\pi)^\leqslant$ with $\chi_\infty=\varepsilon_\infty|\cdot|_\infty^k$ and all $\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ we have*

$$\left(\frac{G(\chi)\,P_\mu(k)}{\Omega(\pi,\pi',\varepsilon(\chi_\infty))}\frac{L(\pi\otimes\chi,0)}{L(\pi'^\vee\otimes\chi,0)}\right)^\sigma=\frac{G(\chi^\sigma)\,P_\mu(k)}{\Omega(\pi^\sigma,\pi'^\sigma,\varepsilon(\chi_\infty))}\frac{L(\pi^\sigma\otimes\chi^\sigma,0)}{L((\pi'^\vee)^\sigma\otimes\chi^\sigma,0)}.$$

Here, $P_\mu(k)\in\mathbb{C}$ only depends on k and μ (i.e., it is independent of π,π') and $\varepsilon(\chi_\infty)=\varepsilon_\infty\mathrm{sgn}^k$.

(2) $\mathrm{Crit}(\pi)\subset\mathrm{Crit}(\pi'^\vee)$.

Obviously, Theorem 1.1 follows from Theorem 1.2 using induction over n with the previously known cases $n=1,2$ of Theorem 1.1 serving as starting point for the induction.

Remark 2. Theorem 1.2 (and, hence, Theorem 1.1) is valid only under a certain non-vanishing assumption (cf. below).

2. Relation to Cohomology of locally symetric spaces

We set $S_n(K)=\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K\,\mathrm{SO}_n(\mathbb{R})Z_n(\mathbb{R})^0$ and $F_n(K)=\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K\,\mathrm{SO}_n(\mathbb{R})$, whence, a natural map $p:F_n(K)\to S_n(K)$. We denote by $\overline{S}_n(K)$ the Borel–Serre compactification of $S_n(K)$. $\overline{S}_n(K)=\cup_{P/\sim}\partial_P S_n(K)$ is a union of faces corresponding to conjugacy classes of rational parabolic subgroups P of GL_n. $\mathcal{M}_\mu^\vee$ denotes the locally constant sheaf on $S_n(K)$ and on $\overline{S}_n(K)$ attached to the finite dimensional representation $M_\mu^\vee$. We set $H^\bullet(S_n,\mathcal{M}_\mu^\vee)=\lim_K H^\bullet(S_n(K),\mathcal{M}_\mu^\vee)$. By assumption, the representation π_f embeds into cohomology

$$\pi_f\hookrightarrow H^{b_n}\big(S_n,\mathcal{M}_\mu^\vee\big).\tag{1}$$

Here, b_n is the lowest degree, in which cuspidal cohomology occurs and π_f occurs with multiplicity 1.

On the other hand, the cohomology of the face $\partial_P S_{n-1}(K)$ attached to the parabolic subgroup $P\leqslant\mathrm{GL}_{n-1}$ of type $(n-2,1)$ with coefficients in $\mathcal{M}_\lambda$ decomposes

$$H^i(\partial_P S_{n-1}, \mathcal{M}_\lambda) = \bigoplus_{w\in W^P} \mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)}\big(H^{i-\ell(w)}(S_{n-2}, \mathcal{M}_{w\cdot\lambda|_{T_{n-2}}}) \otimes H^0(S_1, \mathcal{M}_{w\cdot\lambda|_{\mathbb{G}_m}})\big).$$

Let $\pi' \in \mathcal{C}\mathrm{oh}(\mathrm{GL}_{n-2})$ and $\chi \in \mathrm{Crit}(\pi)$. Our choice of λ implies that $\pi'_f \otimes \chi_f$ embeds into the cohomology of $S_{n-2} \times S_1$ with coefficients in $\mathcal{M}_{\widehat{w}\cdot\lambda|_{T_{n-2}}} \otimes \mathcal{M}_{\widehat{w}\cdot\lambda|_{\mathbb{G}_m}}$. Since $b_{n-1} = b_{n-2} + \ell(\widehat{w})$ we obtain a map

$$\mathrm{Ind}_P^{\mathrm{GL}_{n-1}} \pi'_f \otimes \chi_f \hookrightarrow H^{b_{n-1}}(\partial_P S_{n-1}, \mathcal{M}_\lambda) \stackrel{\mathrm{Eis}}{\to} H^{b_{n-1}}(S_{n-1}, \mathcal{M}_\lambda). \tag{2}$$

The last arrow is given by Eisenstein summation (cf. [3,6]). Since $\lambda \preceq \mu$ we know that $M_\lambda \hookrightarrow M_\mu|_{\mathrm{GL}_{n-1}}$ and we obtain a diagram

$$\begin{array}{ccccc} H_c^{b_n}(S_n, \mathcal{M}_\mu^\vee) & & H^{b_{n-1}}(S_{n-1}, \mathcal{M}_\lambda) & & \\ \downarrow i^* & & \downarrow p^* & & \\ H^{b_n}(F_{n-1}, \mathcal{M}_\mu^\vee|_{\mathrm{GL}_{n-1}}) & \otimes & H^{b_{n-1}}(F_{n-1}, \mathcal{M}_\lambda) & \xrightarrow{\int w\cup\omega'} & H^{\dim F_{n-1}}(F_{n-1}, \underline{\mathbb{C}}) = \mathbb{C}, \end{array} \tag{3}$$

where $\mathcal{M}_\mu^\vee|_{\mathrm{GL}_{n-1}}$ is the sheaf attached to $M_\mu^\vee|_{\mathrm{GL}_{n-1}}$ and $i : F_{n-1}(K) \to S_n(K)$, $g \mapsto \mathrm{diag}(g, 1)$ is the inclusion. Using the description of the cohomology via automorphic forms and combining the *method of Zeta-integrals* (cf. [5]) with the *method of Langlands–Shahidi* (cf. [7]) we are able to compute the pairing defined in (3): there are classes $\omega_\pi \in H^{b_n}(S_n, \mathcal{M}_\mu^\vee)(\pi_f)$ and $\omega_{\mathrm{Eis}\pi'\otimes\chi} \in H_{\mathrm{Eis}}^{b_{n-1}}(S_{n-1}, \mathcal{M}_\lambda)$ such that

$$\sum_u \int i^* r_u^* \omega_\pi \cup p^* \omega_{\mathrm{Eis}\pi'\otimes\chi} = P_\mu(k)\, \frac{L(\pi\otimes\chi, 0)}{L(\pi'\otimes\chi, 0)}. \tag{4}$$

Here, $u \in N_P(\mathbb{A}_f)$ runs over a finite set of unipotent matrices and r_u denotes right translation by u. $P_\mu(k)$ is the quotient of the local factor at infinity of $\sum_u \int i^* r_u^* \omega_\pi \cup p^* \omega_{\mathrm{Eis}\pi'\otimes\chi}$ by $L(\pi_\infty \otimes \chi_\infty, 0)/L(\pi'_\infty \otimes \chi_\infty, 0)$. The choice of the local components at infinity of ω_π and $\omega_{\mathrm{Eis}\pi'\otimes\chi}$ is already determined by the coefficient systems M_μ and M_λ and we de not know whether for these choices $P_\mu(k) = 1$ or at least $P_\mu(k) \neq 0$. Thus, we have to make the

Assumption. $P_\mu(k) \neq 0$.

In case n odd we have computed the SO_n resp. SO_{n-1}-types supporting the cohomology classes ω_π resp. $\omega_{\mathrm{Eis}\pi'\otimes\chi}$ and we verified that they allow for non-vanishing of $P_\mu(k)$. Using ideas of [3] we are able to compute the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the cohomology: we find that after dividing by an appropriate complex number $\Omega(\pi, \pi', \varepsilon(\chi_\infty))$ the left-hand side in (4) behaves equivariantly with respect to the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, which finally yieds Theorem 1.2. As a last remark we note that the assumption that μ be regular perhaps can be circumvented by allowing more general parabolic subgroups $P \leqslant \mathrm{GL}_{n-1}$.

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