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Partial Differential Equations

Generalized scattering phases for asymptotically hyperbolic manifolds

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Abstract

We prove asymptotic expansions of generalized scattering phases asssociated to pairs of Laplacians, for a class of noncompact manifolds with infinite volume and negative curvature near infinity. We use one of these expansions to define relative determinants which appear naturally in this context. ***To cite this article: J.-M. Bouclet, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Phases de diffusion generalisées pour des variétés asymptotiquement hyperboliques. On démontre des développements asymptotiques de phases de diffusions généralisées associées à des couples de Laplaciens, pour une classe de variétés non compactes, de volume infini et à courbure négative près de l'infini. On utilise un de ces développements pour définir des déterminants relatifs qui interviennent de façon naturelle dans ce contexte. ***Pour citer cet article : J.-M. Bouclet, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

In this Note we display recent results [3] of spectral analysis on asymptotically hyperbolic manifolds of dimension $n \geqslant 2$. Our definition of such a manifold (X, G) is the the one of [8]: we assume the existence of a relatively compact set $\mathcal{U}$ such that $X \setminus \mathcal{U}$ is isometric to $(0, \varepsilon)_x \times Y$, equipped with a metric of the form

$$\frac{\mathrm{d}x^2 + h(x)}{x^2}$$

with $h(\cdot)$ a family of metrics on the compact manifold (not necessarily connected) Y, depending smoothly on $x \in [0, \varepsilon)$. Note that X is of infinite volume and that the smoothness of h at $x = 0$ reflects the fact that it is a perturbation of $h(0)$. For any such G, we can consider the related volume form $\mathrm{d}\mu_G$ and the Laplacian Δ_G, so that Δ_G is essentially self-adjoint from $C_0^\infty(X)$.

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If we are given two such metrics G^0 and G^1, i.e., of the form $(\mathrm{d}x^2 + h^i(x))/x^2$ on $(0,\varepsilon) \times Y$, $i = 0, 1$, we will say that they *coincide at infinity* if

$$h^1(0) = h^0(0).$$

Our aim is to *compare* the associated Laplacians. More precisely, we will consider the operators

$$H_0 = \Delta_{G^0}, \qquad H_1 = U^{-1}\Delta_{G^1}U$$

which are both essentially self-adjoint on $L^2(X) := L^2(X, \mathrm{d}\mu_{G^0})$ by choosing $U = (\mathrm{d}\mu_{G^0}/\mathrm{d}\mu_{G^1})^{1/2}$. There is a large literature on the scattering theory of such operators, especially on scattering poles or scattering matrices for which we quote [1,4–6,8,10]; our goal is to define some new tools which appear naturally in this framework and whose lack lead some authors to technical restrictions.

We were for instance motivated by the recent paper [1] where Borthwick, Judge and Perry study the *relative determinant* $\mathrm{Det}_{\mathbf{1}}(z)$ (see below for a definition) associated to a pair of Laplacians on a hyperbolic surface. Since they work on a surface, they can do natural assumptions on their operators, namely that the coefficients of $H_1 - H_0$ are $\mathcal{O}(x^2)$ near $x = 0$, allowing them to define $\mathrm{Det}_{\mathbf{1}}(z)$ thanks to the relative zeta function [11], using Birman–Krein's theory. However this method fails in higher dimensions: in that case, one need to do artificial assumptions on $H_1 - H_0$, which excludes metrics which coincide at infinity, in order to use Birman–Krein's theory. Recall that this theory [12] states that, for any pair of self-adjoint operators A_0, A_1 such that $(A_1 + \mathrm{i})^{-N} - (A_0 + \mathrm{i})^{-N}$ is trace class, there is a function $\xi_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$\mathrm{tr}\big(f(A_1) - f(A_0)\big) = \langle \xi_1', f\rangle, \quad f \in \mathcal{S}(\mathbb{R}),$$

where $\langle\cdot,\cdot\rangle$ is the pairing between $\mathcal{S}$ and $\mathcal{S}'$. If $A_1 = H_1$ and $A_0 = H_0$, the trace class condition depends on the decay of $H_1 - H_0$ near infinity, i.e., near $x = 0$: it is satisfied if the coefficients of $H_1 - H_0$ are $\mathcal{O}(x^{n-1+\delta})$ for some $\delta > 0$. But the latter is to strong if one only requires that G^0 and G^1 coincide at infinity, since $H_1 - H_0 = \mathcal{O}(x)$ in that case. Nevertheless, if $G^0 - G^1 = \mathcal{O}(x^2)$, which happens in many interesting applications, then $H_1 - H_0 = \mathcal{O}(x^2)$ fulfills the trace class condition in dimension $n = 2$, and this is precisely the framework of [1].

The first purpose of this Note is to explain how to define relative determinants in the general case. We will proceed as in [2], where long range perturbations of the Euclidian Laplacian were considered, using Koplienko's regularization [9] which consists in replacing $f(H_1) - f(H_0)$ by

$$\big[f(H_\varepsilon)\big]_{\mathbf{p}} := f(H_1) - \sum_{j=0}^{p-1} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\varepsilon^j} f(H_\varepsilon)_{|\varepsilon=0}, \quad H_\varepsilon = H_0 + \varepsilon(H_1 - H_0)$$

for a suitable p. Before stating our first theorem, we introduce two other notations; we define the volume density $\mathrm{d}\mu_\varepsilon$ naturally associated with the principal symbol p_ε of H_ε, that is the density associated with the riemannian metric defined by p_ε in the fibers of T^*X. In particular, on $\{x < \varepsilon\}$, we have $\mathrm{d}\mu_\varepsilon/\mathrm{d}\mu_{G^0} = \det(1 + \varepsilon h^1(x)^{-1}h^0(x))^{-1/2}$. We can then define

$$[\mathrm{d}\mu_\varepsilon]_{\mathbf{p}} = \mathrm{d}\mu_1 - \sum_{j<p} j!^{-1}\partial_\varepsilon^j\, \mathrm{d}\mu_{\varepsilon|\varepsilon=0}.$$

Note that both $[f(H_\varepsilon)]_{\mathbf{p}}$ and $[\mathrm{d}\mu_\varepsilon]_{\mathbf{p}}$ are independent of ε; these notations simply reflect the fact that they are defined by Taylor expansions of functions of ε.

Theorem 1. *Let G^0, G^1 be two asymptotically hyperbolic metrics which coincide at infinity. Then*

(i) *for all $p \geqslant n$ and all f in the Schwartz space, $[f(H_\varepsilon)]_{\mathbf{p}}$ is trace class. Furthermore there exists a unique $\xi_{\mathbf{p}} \in \mathcal{S}'(\mathbb{R})$ which vanishes on $(-\infty, 0)$ and such that*

$$\langle \xi_{\mathbf{p}}', f\rangle = \mathrm{tr}\big[f(H_\varepsilon)\big]_{\mathbf{p}}, \quad f \in \mathcal{S}(\mathbb{R}).$$

We call $\xi_{\mathbf{p}}$ the generalized scattering phase of order p.

(ii) *The distribution $\xi_{\mathbf{p}}$ is a continuous function on $((n-1)^2/4, \infty)$.*

(iii) *The Laplace transform of $\xi_{\mathbf{p}}'$ has a full asymptotic expansion as $t \downarrow 0$:*

$$\operatorname{tr}\bigl[\mathrm{e}^{-tH_\varepsilon}\bigr]_{\mathbf{p}} \sim t^{-n/2} \sum_{j \geqslant 0} a_j t^j, \quad \textit{with } a_0 = \Gamma\left(\frac{n}{2}+1\right)(2\pi)^{-n}\omega_n \int_X [\mathrm{d}\mu_\varepsilon]_{\mathbf{p}},$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$.

This first theorem is a regularized form of the usual heat expansion on compact manifolds. It was proved in the asymptotically Euclidian case on $\mathbb{R}^n$ in [2], and it is new for asymptotically hyperbolic manifolds. Using the methods of [2], we can now define the associated determinants of order p:

Corollary 1. *For all $z \notin [0, \infty)$, the regularized zeta function $\zeta_z(s) := \operatorname{tr}[(H_\varepsilon - z)^{-s}]_{\mathbf{p}}$ is well defined for $\operatorname{Re}(s) \gg 1$ and has a meromorphic continuation to the whole complex plane $\mathbb{C}_s$, regular at $s = 0$. Thus the following regularized determinant*

$$\mathrm{Det}_{\mathbf{p}}(z) := \exp\bigl(-\zeta_z'(0)\bigr)$$

is defined and is holomorphic on $\mathbb{C} \setminus [0, \infty)$.

In dimension $n = 2$, with $H_1 - H_0 = \mathcal{O}(x^2)$, the definition of $\mathrm{Det}_{\mathbf{1}}(z)$ of [11,1] is the above one with $p = 1$. There are several reasons justifying that $\mathrm{Det}_{\mathbf{p}}(z)$ is indeed a determinant, for $p \geqslant 1$. For instance, it is shown in [2] that if $H_\varepsilon = -\Delta + \varepsilon V$ on $\mathbb{R}^n_y$, with a long range potential $V(y) = \mathcal{O}(\langle y \rangle^{-\delta})$ for $\delta > n/p$, then

$$\mathrm{Det}_{\mathbf{p}}(z) = \det_{\mathbf{p}}\bigl(1 + V(H_0 - z)^{-1}\bigr)$$

with $\det_{\mathbf{p}}(1 + A)$ the usual Fredholm determinant defined for A in the Schatten class S_p [12].

Our next theorem deals with the pointwise behavior of $\xi_{\mathbf{p}}(\lambda)$. This result is a natural extension of the one announced in [5] where Froese and Hislop show that $\xi_{\mathbf{1}}(\lambda) = \mathcal{O}(\lambda^{n/2})$, for $\lambda \uparrow \infty$. Here again, the use of $\xi_{\mathbf{1}}$ prevent them from considering the natural case $H_1 - H_0 = \mathcal{O}(x)$, as explained before Theorem 1. The introduction of $\xi_{\mathbf{p}}$ will allow us to do so.

We have no room to explain how to prove pointwise expansions of $\xi_{\mathbf{p}}$, however we point out that the proof of the next theorem is based on the introduction of Isozaki–Kitada method [7,2] for asymptotically hyperbolic manifold. The interest of this method is to construct a good microlocal long time approximation of $\mathrm{e}^{-\mathrm{i}tH_\varepsilon/\lambda^{1/2}}$ near infinity. This approximation depends on the geometry of X near infinity and is of special interest for operators $H_0,\ H_1$ which are *noncompactly supported* perturbations of eachothers. This means that its main interest is to deal with the difference between H_0 and H_1 near infinity. Note moreover that, in scattering theory, one usually considers the operator H_1 as a perturbation of H_0 which is viewed as a model operator. Unlike the case of $\mathbb{R}^n$ where $-\Delta$ is a canonical choice of model operator, here we have many possible choices for H_0. Since the manifold can be arbitrary on a large compact set, we will choose an operator H_0 which differs from H_1 only in the neighborhood of infinity. For instance, as the curvature associated with G^1 is asymptotically -1, one can consider an operator H_0 associated with a metric of constant curvature near infinity. The latter can be of interest in view of the recent result of [4].

Theorem 2. *Let G^1 be an asymptotically hyperbolic metric on X. Assume moreover that it is nontrapping* (*i.e., that for any compact set $K \subset S^*X$, there exists T such that $\phi^t(K) \cap K = \emptyset$ for $|t| \geqslant T$, ϕ^t being the geodesic flow*)*. Then, for any G^0 which coincides with G^1 at infinity and outside a neighborhood of infinity, $\xi_{\mathbf{p}}$ has a complete asymptotic expansion as $\lambda \uparrow \infty$*

$$\xi_{\mathbf{p}}(\lambda) \sim \lambda^{n/2} \sum_{j \geqslant 0} b_j \lambda^{-j}, \quad b_0 = (2\pi)^{-n} \omega_n \int_X [\mathrm{d}\mu_\varepsilon]_{\mathbf{p}}.$$

This theorem can be improved in several ways, which we explain below, however it shows anyway a much precised result than the estimate of Froese–Hislop and its main interest is that we can consider natural conditions on $G^1 - G^0$ at infinity. Furthermore, there are very few results on scattering phases in this context [5,6] and ours seems to be the first example of full asymptotic expansion of a scattering phase, for negatively curved manifolds obtained, for instance, by metric perturbation of the hyperbolic space $\mathbb{H}^n$.

The main improvement of this theorem would be to relax the nontrapping condition. In this case, one only expects to get a Weyl formula of the form $\xi_{\mathbf{p}}(\lambda) = b_0 \lambda^{n/2} + \mathcal{O}(\lambda^{(n-1)/2})$. This will be proved elsewhere, using a refinement of the constructions of [3] combined with [4]. One could also relax the condition that H_1 and H_0 should coincide on a large compact set. This would use the analysis of [3] combined with extra technical modifications, such as some improved propagation estimates coming from Mourre theory, which will be proved separately.

References

[1] D. Borthwick, C. Judge, P.A. Perry, Determinants of Laplacians and isopolar metrics on surfaces of infinite area, Duke Math. J. 118 (1) (2003) 61–102.

[2] J.M. Bouclet, Spectral distributions for long range perturbations, J. Funct. Anal., doi:10.1016/j.jfa.2003.07.005, in press.

[3] J.M. Bouclet, Generalized scattering phase for asymptotically hyperbolic manifolds, preprint available at http://arXiv.org/abs/math.SP/0312381.

[4] C. Cuevas, G. Vodev, Sharp bounds on the number of resonances for conformally compact manifolds with constant negative curvature near infinity, Comm. Partial Differential Equations 28 (9–10) (2003) 1685–1704.

[5] R. Froese, P.D. Hislop, On the distribution of resonances for some asymptotically hyperbolic manifolds, in: Journées EDP Nantes, exp. VII, 2000.

[6] L. Guillopé, M. Zworski, Scattering asymptotics for Riemann surfaces, Ann. of Math. 145 (1997) 597–660.

[7] H. Isozaki, H. Kitada, Modified wave operators with time independant modifiers, J. Fac. Sci. Univ. Tokyo Sect. IA 32 (1985) 77–104.

[8] M.S. Joshi, A. Sá Barreto, Inverse scattering on asymptotically hyperbolic manifolds, Acta Math. 184 (2000) 41–86.

[9] L.S. Koplienko, Trace formula for nontrace-class perturbations, Siberian Math. J. 25 (1984) 62–71.

[10] R. Mazzeo, R.B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (2) (1987) 260–310.

[11] W. Müller, Relative zeta functions, relative determinants and scattering theory, Commun. Math. Phys. 192 (2) (1998) 309–347.

[12] D. Yafaev, Mathematical Scattering Theory. General Theory, in: Transl. Math. Monographs, vol. 105, American Mathematical Society, Providence, RI, 1992.