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A multi-domain method for solving numerically multi-scale elliptic problems

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Abstract

In this paper we present a family of iterative methods to solve numerically second order elliptic problems with multi-scale data using multiple levels of grids. These methods are based upon the introduction of a Lagrange multiplier to enforce the continuity of the solution and its fluxes across interfaces. This family of methods can be interpreted as a mortar element method with complete overlapping domain decomposition for solving numerically multi-scale elliptic problems. ***To cite this article: R. Glowinski et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une méthode multi-domaines pour résoudre numériquement des problèmes elliptiques multi-échelles. Dans cette Note nous présentons une famille de méthodes itératives pour résoudre numériquement des problèmes elliptiques du deuxième ordre à données multi-échelles utilisant plusieurs niveaux de grilles. Ces méthodes sont basées sur l'introduction d'un multiplicateur de Lagrange pour imposer la continuité de la solution et de ses flux à travers les interfaces. Ces méthodes peuvent être interprétées comme des méthodes de décomposition de domaines avec recouvrement total, de type mortier, pour résoudre numériquement des problèmes elliptiques multi-échelles. ***Pour citer cet article : R. Glowinski et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, regular domain of $\mathbb{R}^2$ with boundary Γ and let $\omega \Subset \Omega$ be another open, regular domain with boundary γ, which is, in practice, "much smaller" than Ω. If $f \in L^2(\Omega)$ is formed by the sum of two functions $f_1, f_2 \in L^2(\Omega)$ with $\mathrm{supp}(f_2) \subset \omega$ ($f = f_1 + f_2$), we consider the following elliptic problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

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whose weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega f_1 \phi \, \mathrm{d}x + \int_\omega f_2 \phi \, \mathrm{d}x, \quad \forall \phi \in H_0^1(\Omega). \tag{2}$$

Keeping in mind that Ω and ω differ by several scale factors, the discretization of (2) by a finite element method cannot be done using grids with the same resolution on $\overline{\Omega}$ and $\overline{\omega}$. Therefore we introduce two auxiliary problems.

Let $\lambda \in H^{-1/2}(\gamma)$ and let $v \in H_0^1(\Omega)$ be such that

$$\int_\Omega \nabla v \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega f_1 \phi \, \mathrm{d}x + \langle \lambda, \phi \rangle, \quad \forall \phi \in H_0^1(\Omega), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality $H^{-1/2}(\gamma),\ H^{1/2}(\gamma)$. Actually, v is the solution (in the weak sense) of the following problem

$$\begin{cases} -\Delta v = f_1 & \text{in } \Omega \setminus \gamma, \\ v = 0 & \text{on } \Gamma, \\ [v] = 0 & \text{on } \gamma, \\ \left[\dfrac{\partial v}{\partial n} \right] = -\lambda & \text{on } \gamma, \end{cases} \tag{4}$$

where $[\psi]$ denotes the jump of ψ on γ, when choosing **n** as the unit normal vector outward to the domain ω on γ.

Henceforth, we define $w \in H^1(\omega)$ such that $w = v$ on γ and

$$\int_\omega \nabla w \cdot \nabla \phi \, \mathrm{d}x = \int_\omega (f_1 + f_2) \phi \, \mathrm{d}x, \quad \forall \phi \in H_0^1(\omega), \tag{5}$$

whose strong formulation is

$$\begin{cases} -\Delta w = f_1 + f_2 & \text{in } \omega, \\ w = v & \text{on } \gamma. \end{cases} \tag{6}$$

Since $\mathrm{supp}(f_2) \subset \omega$, and with regard to (4) and (6), we see immediately that if λ is chosen such that

$$\frac{\partial v^+}{\partial n} = \frac{\partial w}{\partial n} \quad \text{on } \gamma, \tag{7}$$

where v^+ is the restriction of v to $\Omega \setminus \overline{\omega}$, then the solution u of (1) or (2) is given by $u = v^+$ on $\Omega \setminus \overline{\omega}$ and $u = w$ on ω. Therefore, defining the operator

$$T : H^{-1/2}(\gamma) \to H^{-1/2}(\gamma)$$

by

$$T\lambda = \frac{\partial w}{\partial n} - \frac{\partial v^+}{\partial n}, \tag{8}$$

it is sufficient to find λ such that $T\lambda = 0$. We can verify that T is the affine operator given by

$$T\lambda = \lambda + \frac{\partial \overline{w}}{\partial n} - \frac{\partial \bar{v}^+}{\partial n}, \tag{9}$$

where $\bar{v}$ satisfies $-\Delta \bar{v} = f_1$ in Ω with $\bar{v} = 0$ on Γ and $\overline{w}$ satisfies $-\Delta \overline{w} = f_1 + f_2$ in ω with $\overline{w} = \bar{v}$ on γ.

Thus it is trivial that $\lambda = \partial \bar{v}^+/\partial n - \partial \overline{w}/\partial n$ when $T\lambda = 0$.

Let us suppose now that $\overline{\Omega}$ is a polygonal domain and let $\mathcal{T}_H$ be a triangulation of $\overline{\Omega}$ with triangles K. We also assume that the boundary γ coincides with the union of edges of triangles K of $\mathcal{T}_H$. Let $\mathcal{T}_h$ be a triangulation of $\overline{\omega}$ that is not necessarily nested with $\mathcal{T}_H$. Moreover, we introduce

$$V_H = \{\phi \in C^0(\overline{\Omega}) \colon \phi|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_H,\ \phi = 0 \text{ on } \Gamma\}$$

and

$$W_h = \{\phi \in C^0(\overline{\omega}) \colon \phi|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(K)$ denotes the set of polynomials of degree 1 on the triangle K. We also define

$$\Lambda_H = \{\mu \in C^0(\gamma) \colon \exists \tilde{\mu} \in V_H \text{ satisfying } \mu = \tilde{\mu} \text{ on } \gamma\}.$$

Now, the idea is to define an operator $T_H : \Lambda_H \to \Lambda_H$ approximating appropriately the operator T, and then to search for $\lambda_H \in \Lambda_H$ such that $T_H\lambda_H = 0$. Thus, an approximation of the solution u of problem (1) is given by u_h with $u_h = v_H^+$ in $\Omega \setminus \overline{\omega}$ and $u_h = w_h$ in ω, where v_H^+ and w_h are approximations of v^+ and w in V_H and W_h respectively. More precisely, v_H^+ is the restriction of v_H to $\Omega \setminus \overline{\omega}$, where $v_H \in V_H$ satisfies:

$$\int_\Omega \nabla v_H \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega f_1\phi \,\mathrm{d}x + \int_\gamma \lambda_H\phi \,\mathrm{d}s, \quad \forall \phi \in V_H. \tag{10}$$

Besides, $w_h \in W_h$ is such that $w_h = R_h v_H$ on γ and

$$\int_\omega \nabla w_h \cdot \nabla\phi \,\mathrm{d}x = \int_\omega (f_1 + f_2)\phi \,\mathrm{d}x, \quad \forall \phi \in W_h \cap H_0^1(\omega), \tag{11}$$

where R_h is the classical interpolation operator to the space W_h.

We propose three ways to construct T_H.

P1: If λ_H is given, we solve (10) and (11) and, drawing our inspiration from (8), we define $\delta_H \in \Lambda_H$ such that

$$\int_\gamma \delta_H\mu \,\mathrm{d}s = \int_\gamma \left(\frac{\partial w_h}{\partial n} - \frac{\partial v_H^+}{\partial n}\right)\mu \,\mathrm{d}s, \quad \forall \mu \in \Lambda_H. \tag{12}$$

Thus we are led to the definition $T_H\lambda_H = \delta_H$.

P2: The quantity $(\partial w_h/\partial n - \partial v_H^+/\partial n)$ is piecewise constant on the boundary γ. To make it more regular, we can consider the L^2-projections Dv_H^+ and Dw_h of ∇v_H^+ and ∇w_h to the spaces $V_H^+ = \{\phi \in C^0(\overline{\Omega} \setminus \omega) \colon \phi|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_H,\ K \subset \overline{\Omega} \setminus \omega\}$ and W_h respectively. Thus $\delta_H = T_H\lambda_H$ is given by

$$\int_\gamma \delta_H\mu \,\mathrm{d}s = \int_\gamma \left(Dw_h - Dv_H^+\right) \cdot \mathbf{n}\,\mu \,\mathrm{d}s, \quad \forall \mu \in \Lambda_H. \tag{13}$$

P3: Still drawing our inspiration from relation (8), for the definition of $\delta_H \in \Lambda_H$, we approximate the quantity $\int_\gamma \frac{\partial w_h}{\partial n}\mu \,\mathrm{d}s$ and $-\int_\gamma \frac{\partial v_H^+}{\partial n}\mu \,\mathrm{d}s$ by $\int_\omega \nabla w_h \cdot \nabla\tilde{\mu} \,\mathrm{d}x - \int_\omega (f_1 + f_2)\tilde{\mu} \,\mathrm{d}x$ and $\int_{\Omega\setminus\overline{\omega}} \nabla v_H^+ \cdot \nabla\tilde{\mu} \,\mathrm{d}x - \int_{\Omega\setminus\overline{\omega}} f_1\tilde{\mu} \,\mathrm{d}x$ respectively, $\forall \mu \in \Lambda_H$, where $\tilde{\mu}$ is an extension of μ in V_H. Taking into account the relation (10), we have, $\forall \mu \in \Lambda_H$,

$$\int_\gamma \delta_H\mu \,\mathrm{d}s = \int_\gamma \lambda_H\mu \,\mathrm{d}s + \left(\int_\omega \nabla w_h \cdot \nabla\tilde{\mu} \,\mathrm{d}x - \int_\omega (f_1 + f_2)\tilde{\mu} \,\mathrm{d}x\right) - \left(\int_\omega \nabla v_H^- \cdot \nabla\tilde{\mu} \,\mathrm{d}x - \int_\omega f_1\tilde{\mu} \,\mathrm{d}x\right), \tag{14}$$

where v_H^- is the restriction of v_H to $\overline{\omega}$. Relation (14) doesn't depend on the actual extension and we choose here $\tilde{\mu}$ the extension of μ such that $\tilde{\mu}|_K = 0,\ \forall K \in \mathcal{T}_H,\ K \cap \gamma = \emptyset$.

Problem P3 can be interpreted as a mortar element formulation with complete overlapping domain decomposition (see [2]). For error estimates, see [1] and [3].

We propose two methods to solve $T_H\lambda_H = 0$.

M1: Fixed point method. Drawing inspiration from (9), we write in all cases $T_H = I - (I - T_H)$ where I is the identity in Λ_H. Clearly $(I - T_H)$ has to be "close" to a constant operator and if $\lambda_H \in \Lambda_H$ is such that $T_H\lambda_H = 0$, then $\lambda_H = (I - T_H)\lambda_H$. In this case the method used for solving $T_H\lambda_H = 0$ becomes:

$$\lambda_H^{k+1} = \lambda_H^k - T_H\lambda_H^k, \quad k = 0, 1, 2, \ldots, \tag{15}$$

after initialization of $\lambda_H^0 \in \Lambda_H$.

The fixed point method M1 for problem P3 can be viewed as an efficient iterative method, based on a suitable reformulation of the problem in terms of a Steklov–Poincaré interface equation (see [5,6]).

M2: Minimization method. In this case we find $\lambda_H^* \in \Lambda_H$ such that

$$\mathcal{J}(\lambda_H^*) \leqslant \mathcal{J}(\lambda_H), \quad \forall\lambda_H \in \Lambda_H, \tag{16}$$

where the cost function $\mathcal{J}(\cdot)$ is defined as

$$\mathcal{J}(\lambda_H) = \frac{1}{2}\int_\gamma \delta_H^2\,\mathrm{d}s. \tag{17}$$

A conjugate gradient algorithm can be used for the solution of (16). With the above described problem P3, where δ_H is defined by (14), the gradient of $\mathcal{J}(\lambda_H)$ can be computed as follows (see [5] for details). First compute $q_h \in W_h$ such that $q_h = 0$ on γ and

$$\int_\omega \nabla q_h \cdot \nabla\phi\,\mathrm{d}x = \int_\omega \nabla\tilde{\delta}_H \cdot \nabla\phi\,\mathrm{d}x, \quad \forall\phi \in W_h \cap H_0^1(\omega), \tag{18}$$

where $\tilde{\delta}_H$ is an extension of δ_H in V_H. Then find $\xi_H \in \Lambda_H$ such that

$$\int_\gamma \xi_H\mu\,\mathrm{d}s = -\int_\omega \nabla q_h \cdot \nabla\tilde{\mu}\,\mathrm{d}x + \int_\omega \nabla\tilde{\delta}_H \cdot \nabla\tilde{\mu}\,\mathrm{d}x, \quad \forall\mu \in \Lambda_H, \tag{19}$$

and $p_H \in V_H$ satisfying

$$\int_\Omega \nabla p_H \cdot \nabla\phi\,\mathrm{d}x = \int_\gamma \xi_H\phi\,\mathrm{d}s + \int_{\Omega\setminus\bar{\omega}} \nabla\tilde{\delta}_H \cdot \nabla\phi\,\mathrm{d}x, \quad \forall\phi \in V_H. \tag{20}$$

Finally, the gradient $g_H = \nabla_\lambda\mathcal{J}(\lambda_H) \in \Lambda_H$ is given as $\int_\gamma g_H\mu\,\mathrm{d}s = \int_\gamma p_H\mu\,\mathrm{d}s,\ \forall\mu \in \Lambda_H$.

Numerical example. We illustrate the above presented algorithm with the following example whose results will be compared with [4] in [5]: Consider the Poisson–Dirichlet problem $-\Delta u = f$ in the domain $\Omega = (-1, 1)^2,\ u = 0$ on its boundary $\partial\Omega$. Take $f = 2k^2\pi^2\cos(k\pi x_1)\cos(k\pi x_2) - 4\eta\chi(r)\exp(\frac{1}{\varepsilon^2})\exp(\frac{-1}{|\varepsilon^2-r^2|})\frac{r^2+r^4-\varepsilon^4}{|\varepsilon^2-r^2|^4}$ where $r = \sqrt{x_1^2 + x_2^2}$ and $\chi(r) = 1$ if $r \leqslant \varepsilon,\ \chi(r) = 0$ if $r > \varepsilon;\ k,\ \eta$ and ε are parameters. The exact solution to the problem is given by $u = \cos(k\pi x_1)\cos(k\pi x_2) + \eta\chi(r)\exp(\frac{1}{\varepsilon^2})\exp(\frac{-1}{|\varepsilon^2-r^2|})$. We choose $k = 0.5,\ \eta = 10$ and $\varepsilon = 0.25$ to start with.

Away from the origin $(0, 0)$ the solution is smooth. In a region close to $(0, 0)$ where the solution is peaking, we need to apply a patch with a finer mesh. For the triangulation of Ω, we use a coarse uniform grid with mesh size H. We consider a patch $\omega = (-0.25, 0.25)^2$ with a fine uniform triangulation of size h. We consider the case where the fine triangulation is nested in the coarse one. The mesh sizes H and h are chosen in a way that the origin $(0, 0)$ is always a grid point. All numerical quadratures are done using schemes which are

Table 1
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 4$ and $\varepsilon = 0.25$

Tableau 1
Erreur relative en norme L^2, H^1 et L^∞ et convergence de l'algorithme pour H décroissant avec rapport $H/h = 4$ et $\varepsilon = 0.25$

H	1/4				1/8				1/16				1/32			
Meth.	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞
P1–M1	30	1.75E−1	4.14E−2	2.58E−2	13	7.06E−2	1.70E−2	1.02E−2	27	3.00E−2	7.63E−3	4.27E−3	20	1.36E−2	3.65E−3	1.91E−3
P2–M1	28	1.82E−1	4.15E−2	2.40E−2	15	8.08E−2	1.85E−2	1.14E−2	28	3.63E−2	8.53E−3	5.31E−3	20	1.73E−2	4.17E−3	2.65E−3
P3–M1	6	2.89E−2	2.54E−2	4.27E−3	5	7.23E−3	1.16E−2	1.13E−3	6	1.79E−3	5.59E−3	2.77E−4	4	4.47E−4	2.77E−3	6.83E−5
P3–M2	10	2.89E−2	2.54E−2	4.27E−3	7	7.25E−3	1.16E−2	1.13E−3	6	1.79E−3	5.59E−3	2.77E−4	6	4.47E−4	2.77E−3	6.82E−5

Table 2
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 4$ and $\varepsilon = 0.5$

Tableau 2
Erreur relative en norme L^2, H^1 et L^∞ et convergence de l'algorithme pour H décroissant avec rapport $H/h = 4$ et $\varepsilon = 0.5$

H	1/4				1/8				1/16				1/32			
Meth.	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞	Iter.	L^2	H^1	L^∞
P1–M1	32	3.50E0	1.67E0	1.37E0	20	2.05E0	9.20E−1	8.10E−1	12	9.52E−1	4.29E−1	3.83E−1	10	4.36E−1	1.99E−1	1.77E−1
P2–M1	24	3.88E0	1.69E0	1.43E0	15	2.11E0	9.30E−1	8.04E−1	12	9.61E−1	4.26E−1	3.71E−1	11	4.32E−1	1.94E−1	1.69E−1
P3–M1	6	1.88E−1	3.52E−1	1.27E−1	6	5.76E−2	2.38E−1	6.65E−2	5	1.38E−2	1.22E−1	2.45E−2	5	3.63E−3	6.18E−2	7.39E−3
P3–M2	6	1.88E−1	3.52E−1	1.27E−1	6	5.76E−2	2.38E−1	6.65E−2	7	1.38E−2	1.22E−1	2.45E−2	7	3.63E−3	6.18E−2	7.39E−3

exact for polynomials of degree 2. We compute the numerical approximation of problem (1) on a global fine uniform triangulation with mesh size h which is an extension of the fine triangulation to the domain Ω. In the sequel we call this approximation the reference solution. That way, we minimize the projection errors introduced when comparing the results against the reference solution. We outline here the relative errors of the solution calculated on the reference grid with $h = 1/16,\ 1/32,\ 1/64$ and $1/128$. The relative L^2-norm error gives the values $3.81\text{E}{-2},\ 1.42\text{E}{-2},\ 4.09\text{E}{-3},\ 1.06\text{E}{-3}$; the relative H^1-norm error yields $1.43\text{E}{-1},\ 4.64\text{E}{-2},\ 1.22\text{E}{-2}$ and $3.09\text{E}{-3}$.

Table 1 indicates the convergence of the algorithm to the reference solution with successive smaller H and fixed ratio $H/h = 4$. Results of definitions P1, P2 and P3 using the fixed point method M1 are depicted as well as definition P3 using the above described conjugate gradient minimization method M2. The stopping criteria for the fixed point algorithm is $|d_n - d_{n-1}|/d_0 < 10^{-3}$ where d_n is the L^2-norm of δ_H at iteration $n,\ n = 1, 2, \ldots$. For the conjugate gradient, the stopping criteria is chosen as 10^{-8}.

All problems provide acceptable results for the correction of the jump of the first derivative. We observe optimal convergence for the definition P3: the asymptotic rate of convergence in h is 2 for the L^2-norm and 1 for the H^1-norm. The fixed point method M1 and the minimization method M2 yield the same solution. The number of iterations depends very little on the mesh size.

We now choose $\varepsilon = 0.5$. In this case the peaking part of f is no more entirely included in the patch ω. Let us first outline the error of the solution calculated on the reference grid with $h = 1/16,\ 1/32,\ 1/64$ and $1/128$. The relative L^2-norm error gives the values $6.31\text{E}{-2},\ 2.12\text{E}{-3},\ 5.38\text{E}{-4},\ 1.35\text{E}{-4}$; the relative H^1-norm error yields $1.77\text{E}{-1},\ 4.63\text{E}{-3},\ 1.17\text{E}{-3}$ and $2.94\text{E}{-4}$.

Table 2 gives the results of the three methods in this case. The approximation given by definitions P1 and P2 is rather poor. The result of definition P3 approaches the reference solution well. The error degenerates as part of the solution is badly resolved on the coarse grid.

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