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Probability Theory

Some results on the uniqueness of generators of backward stochastic differential equations

Long Jiang [a,b]

[a] *Department of Mathematics, Shandong University, 250100 Jinan, China*
[b] *Department of Mathematics, China University of Mining and Technology, 221008 Xuzhou, China*

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Abstract

It is proved that the generator g of a backward stochastic differential equation (BSDE) can be uniquely determined by the initial values of the corresponding BSDEs with all terminal conditions. The main results also confirm and extend Peng's conjecture. ***To cite this article: L. Jiang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Quelques résultats sur l'unicité des générateurs des équations différentielles stochastiques rétrogrades. L'auteur prouve que le générateur g d'une équation différentielle stochastique rétrograde peut être déterminé uniquement par les valeurs initiales d'équation différentielle stochastique rétrograde correspondante avec toutes les conditions terminales. Le résultat principal confirme et étend le résultat de Peng. ***Pour citer cet article : L. Jiang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Considérons un nombre réel $T>0$. Soit g un générateur d'équation différentielle stochastique rétrograde qui satisfait les hypothèses (A1), (A2) et (A5) (voir Section 2 pour les détails). Pour chaque temps d'arrêt $\tau\leqslant T$ et $\xi\in L^2(\Omega,\mathcal{F}_\tau,P)$, denotons $(y_t^{(\tau,g,\xi)},z_t^{(\tau,g,\xi)})_{t\in[0,\tau]}$ la solution adaptée unique et de carré integrable de l'EDSR

$$y_t=\xi+\int_t^\tau g(s,y_s,z_s)\,\mathrm{d}s-\int_t^\tau z_s\,\mathrm{d}B_s,\quad t\in[0,\tau].$$

Théorème 0.1. *Supposons que les deux générateurs g_1 et g_2 vérifient* (A1), (A2) *et* (A5)*. Alors, les deux conditions suivantes* (i) *et* (ii) *sont équivalentes* :

E-mail address: jianglong@math.sdu.edu.cn (L. Jiang).

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(i) *P-a.s., pour chaque triplet* $(t,y,z)\in[0,T]\times\mathbf{R}\times\mathbf{R}^d$*, on a :* $g_1(t,y,z)=g_2(t,y,z)$ *;*
(ii) *Pour chaque temps d'arrêt* $\tau\leqslant T$*, on a :* $y_0^{(\tau,g_1,\xi)}=y_0^{(\tau,g_2,\xi)}$, $\forall\xi\in L^2(\Omega,\mathcal{F}_\tau,P)$.

Si on suppose que g_1 *et* g_2 *satisfont :* P*-a.s.,* $\forall t\in[0,T]$, $g_1(t,0,0)\equiv 0$, $g_2(t,0,0)\equiv 0$. *Alors,* (i) *est equivalent à la condition suivante* (iii) :

(iii) *Pour chaque* $r\in[0,T]$*, on a :* $y_0^{(r,g_1,\xi)}=y_0^{(r,g_2,\xi)}$, $\forall\xi\in L^2(\Omega,\mathcal{F}_r,P)$.

1. Introduction

It is by now well known (see [6]) that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE in short) of type

$$y_t=\xi+\int_t^T g(s,y_s,z_s)\,\mathrm{d}s-\int_t^T z_s\,\mathrm{d}B_s,\quad 0\leqslant t\leqslant T, \tag{1}$$

providing, for instance, that the function g is Lipschitz in both variables y and z, and that ξ and $(g(s,0,0))_{s\in[0,T]}$ are square integrable; g is said to be the generator of BSDE (1), (T,ξ) are called terminal conditions, and (T,g,ξ) are called standard parameters of BSDE (1). We denote the unique adapted and square integrable solution of BSDE (1) by $(y_t^{(T,g,\xi)},z_t^{(T,g,\xi)})_{t\in[0,T]}$.

In 1997, Peng proposed a conjecture; roughly speaking, the conjecture is:

If a generator g satisfies $g(t,y,0)\equiv 0$, then g is uniquely determined by the initial values $y_0^{(T,g,\xi)}$ for all square integrable ξ. In other words, if two generators g_1, g_2 satisfy $g_1(t,y,0)\equiv 0$, $g_2(t,y,0)\equiv 0$, and $y_0^{(T,g_1,\xi)}=y_0^{(T,g_2,\xi)}$ for every square integrable ξ, then $g_1=g_2$.

Chen [2] confirmed Peng's conjecture, and it will be cited as a proposition.

However, the condition $g(t,y,0)\equiv 0$ is too strong, most generators g do not satisfy it, especially when we apply BSDE theory to European option pricing (see [5]) or recursive utility (see [3]); in these cases the generators of BSDEs often do not satisfy $g(t,y,0)\equiv 0$. Thus it yields a natural question:

Without the hypothesis $g(t,y,0)\equiv 0$, can we prove that g is determined uniquely by the initial values of the corresponding BSDEs with all terminal conditions?

The objective of this paper is to investigate this problem and in Section 3 the author will prove that, under some natural and reasonable assumptions, the answer is 'Yes'.

2. Preliminaries

Firstly let us introduce some notations and assumptions. Let $(\Omega,\mathcal{F},P)$ be a probability space and $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0=0$; let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration generated by this Brownian motion: $\mathcal{F}_t=\sigma\{B_s,s\in[0,t]\}\vee\mathcal{N}$, $t\in[0,T]$, where $\mathcal{N}$ is the set of all P-null subsets.

Let $T>0$ be a given real number. In this paper, we always work in the space $(\Omega,\mathcal{F}_T,P)$, only consider processes indexed by $t\in[0,T]$. For any positive integer n and $z\in\mathbf{R}^n$, $|z|$ denotes its Euclidean norm.

We define the following usual spaces of processes:

$\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbf{R}):=\{\psi$ progressively measurable; $\mathbf{E}[\sup_{0\leqslant t\leqslant T}|\psi_t|^2]<\infty\}$;

$\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^n):=\{\psi$ progressively measurable; $\|\psi\|_2^2=\mathbf{E}[\int_0^T|\psi_t|^2\,\mathrm{d}t]<\infty\}$;

The generator g of a BSDE is a function $g(\omega,t,y,z):\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\mapsto\mathbf{R}$ such that $(g(t,y,z))_{t\in[0,T]}$ is progressively measurable for each (y,z) in $\mathbf{R}\times\mathbf{R}^d$, and g also satisfies some of the following assumptions:

(A1) There exists a constant $K\geqslant 0$, such that P-a.s., we have:

$$\forall t,\forall y_1,y_2,z_1,z_2:\ \big|g(t,y_1,z_1)-g(t,y_2,z_2)\big|\leqslant K\big(|y_1-y_2|+|z_1-z_2|\big).$$

(A2) The process $(g(t,0,0))_{t\in[0,T]} \in \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R})$.
(A3) P-a.s., $\forall t \in [0,T]$, $g(t,0,0) \equiv 0$.
(A4) P-a.s., $\forall (t,y) \in [0,T] \times \mathbf{R}$, $g(t,y,0) \equiv 0$.
(A5) P-a.s., $\forall (y,z) \in \mathbf{R} \times \mathbf{R}^d$, $t \to g(t,y,z)$ is continuous.
(A6) P-a.s., $\forall (y,z) \in \mathbf{R} \times \mathbf{R}^d$, $t \to g(t,y,z)$ is right continuous in $t \in [0,T[$ and left continuous in T.

Remark 1. Assumption (A4) implies (A3), (A3) implies (A2); Assumption (A5) implies (A6).

For each given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, let $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]}$ be the unique square integrable and adapted solution of the BSDE (1), such that $y^{(T,g,\xi)}$ is in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbf{R})$ and $z^{(T,g,\xi)}$ is in $\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ (see [6] for details). Please remember that, provided that g satisfies (A1) and (A2), there exists a unique pair $(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)})_{t\in[0,T]}$ of adapted processes in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbf{R}) \times \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ solving the BSDE (1).

In the sequel, we always assume that g satisfies (A1) and (A2), we often denote the initial value $y_0^{(T,g,\xi)}$ of BSDE (1) by $\mathcal{E}_{g,T}[\xi]$; denote $y_t^{(T,g,\xi)}$ by $\mathcal{E}_{g,T}[\xi|\mathcal{F}_t]$.

Now let us list some basic properties of BSDEs; the following Lemmas 2.1 and 2.2 come from [5, Proposition 2.5, Theorem 2.2], Proposition 2.3 is Peng's conjecture, which was confirmed by [2].

Lemma 2.1. *Let g satisfy* (A1) *and* (A2)*, let $\tau \leqslant T$ be a stopping time, $0 \leqslant t \leqslant s \leqslant T$. Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, then*:

(i) $\mathcal{E}_{g,s}[\mathcal{E}_{g,T}[\xi|\mathcal{F}_s]|\mathcal{F}_t] = \mathcal{E}_{g,T}[\xi|\mathcal{F}_t]$;
(ii) $\mathcal{E}_{g,\tau}[\mathcal{E}_{g,T}[\xi|\mathcal{F}_\tau]|\mathcal{F}_t] = \mathcal{E}_{g,T}[\xi|\mathcal{F}_t]$, *a.s.*, *if* $t \in [0,\tau]$.

Lemma 2.2 (Comparison theorem). *Let g satisfy* (A1) *and* (A2) *and let $X_1, X_2 \in L^2(\Omega, \mathcal{F}_T, P)$.*

(i) (*Monotonicity*): *If $X_1 \geqslant X_2$, a.s., then $\mathcal{E}_{g,T}[X_1] \geqslant \mathcal{E}_{g,T}[X_2]$*;
(ii) (*Strict Monotonicity*): *If $X_1 \geqslant X_2$, a.s., and $P(X_1 > X_2) > 0$, then $\mathcal{E}_{g,T}[X_1] > \mathcal{E}_{g,T}[X_2]$.*

Proposition 2.3. *Let g_1, g_2 satisfy assumptions* (A1), (A4) *and* (A5). *We assume moreover that g_1, g_2 satisfy hypothesis*

(H1) $\mathcal{E}_{g_1,T}[\xi] = \mathcal{E}_{g_2,T}[\xi]$, $\forall \xi \in L^2(\Omega, \mathcal{F}_T, P)$.

Then for $\forall (y,z) \in \mathbf{R} \times \mathbf{R}^d$, we have: P-*a.s.*, $\forall t \in [0,T]$, $g_1(t,y,z) = g_2(t,y,z)$.

3. Main results

Theorem 3.1. *Let two generators g_1, g_2 satisfy assumptions* (A1), (A2) *and* (A5). *Then the following two conditions are equivalent*:

(i) (H2) *For each stopping time $\tau \leqslant T$, we have*: $\mathcal{E}_{g_1,\tau}[\xi] = \mathcal{E}_{g_2,\tau}[\xi]$, $\forall \xi \in L^2(\Omega, \mathcal{F}_\tau, P)$.
(ii) *P-a.s., for each triplet $(t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d$, we have*: $g_1(t,y,z) = g_2(t,y,z)$.

Proof. It is obvious that (ii) $\Rightarrow$ (i). So we only need to prove that (i) $\Rightarrow$ (ii). The main approach of the following proof derives from [4].

For each $\delta > 0$ and $(y,z) \in \mathbf{R} \times \mathbf{R}^d$, we define the following stopping time:

$$\tau_\delta(y,z) := \inf\{t \geqslant 0;\ g_1(t,y,z) \leqslant g_2(t,y,z) - \delta\} \wedge T.$$

Suppose that $g_1,\ g_2$ satisfy (H2). If (ii) does not hold, then we may assume, without loss of generality, that there exists $\delta_0 > 0$ and $(y_0, z_0) \in \mathbf{R} \times \mathbf{R}^d$, such that $P(\{\tau_{\delta_0}(y_0, z_0) < T\}) > 0$. For simplicity, we denote $\tau_{\delta_0}(y_0, z_0)$ by τ_0. For such a triplet (δ_0, y_0, z_0), we study the following forward SDEs defined over the interval $[\tau_0, T]$:

$$-\mathrm{d}Y^1(t) = g_1\big(t, Y^1(t), z_0\big)\,\mathrm{d}t - z_0\,\mathrm{d}B_t, \quad Y^1(\tau_0) = y_0; \tag{2}$$

$$-\mathrm{d}Y^2(t) = g_2\big(t, Y^2(t), z_0\big)\,\mathrm{d}t - z_0\,\mathrm{d}B_t, \quad Y^2(\tau_0) = y_0. \tag{3}$$

Let Y^1, Y^2 denote, respectively, the solutions of forward SDE (2) and (3); clearly $Y^1, Y^2 \in \mathcal{S}^2$. Now we define a new stopping time:

$$\sigma_0 := \inf\left\{t \geqslant \tau_0;\ g_1\big(t, Y^1(t), z_0\big) \geqslant g_2\big(t, Y^2(t), z_0\big) - \frac{\delta_0}{2}\right\} \wedge T.$$

Then we have:

Proposition 3.2. (i) $\sigma_0 = T$ *if* $\tau_0 = T$;
(ii) $\{\tau_0 < \sigma_0\} = \{\tau_0 < T\}$ *and* $P(\{\tau_0 < \sigma_0\}) > 0$;
(iii) $Y^1(\sigma_0) > Y^2(\sigma_0)$ *on* $\{\tau_0 < \sigma_0\}$.

Proof. (i) is obvious. Thanks to assumption (A5), from the continuity of $g_1,\ g_2$ in t, we can conclude that $\{\tau_0 < \sigma_0\} = \{\tau_0 < T\}$, hence we also have $P(\{\tau_0 < \sigma_0\}) > 0$. To prove (iii), we notice that on $\{\tau_0 < \sigma_0\}$, from SDE (2) and (3), we have:

$$Y^1(\sigma_0) - Y^2(\sigma_0) = \int_{\tau_0}^{\sigma_0} \big[g_2\big(t, Y^2(t), z_0\big) - g_1\big(t, Y^1(t), z_0\big)\big]\,\mathrm{d}t \geqslant \int_{\tau_0}^{\sigma_0} \frac{\delta_0}{2}\,\mathrm{d}t = \frac{\delta_0}{2}(\sigma_0 - \tau_0) > 0.$$

The proof of Proposition 3.2 is complete. □

We come back to the proof of Theorem 3.1. Studying the solutions of the following two BSDEs which are corresponding to the above two forward SDE (2) and (3):

$$\overline{Y}^1(t) = Y^1(T) + \int_t^T g_1\big(s, \overline{Y}^1(s), \overline{Z}^1(s)\big)\,\mathrm{d}s - \int_t^T \overline{Z}^1(s)\,\mathrm{d}B_s, \quad t \in [0, T],$$

$$\overline{Y}^2(t) = Y^2(T) + \int_t^T g_2\big(s, \overline{Y}^2(s), \overline{Z}^2(s)\big)\,\mathrm{d}s - \int_t^T \overline{Z}^2(s)\,\mathrm{d}B_s, \quad t \in [0, T],$$

obviously we have:

$$\mathcal{E}_{g_1,T}\big[Y^1(T)|\mathcal{F}_{\tau_0}\big] = \overline{Y}^1(\tau_0) = y_0, \qquad \mathcal{E}_{g_2,T}\big[Y^2(T)|\mathcal{F}_{\tau_0}\big] = \overline{Y}^2(\tau_0) = y_0.$$

By Lemma 2.1, we understand that

$$\mathcal{E}_{g_1,\tau_0}[y_0] = \mathcal{E}_{g_1,\tau_0}\big[\mathcal{E}_{g_1,T}\big[Y^1(T)|\mathcal{F}_{\tau_0}\big]\big] = \mathcal{E}_{g_1,T}\big[Y^1(T)\big] = \mathcal{E}_{g_1,\sigma_0}\big[Y^1(\sigma_0)\big],$$

$$\mathcal{E}_{g_2,\tau_0}[y_0] = \mathcal{E}_{g_2,\tau_0}\big[\mathcal{E}_{g_2,T}\big[Y^2(T)|\mathcal{F}_{\tau_0}\big]\big] = \mathcal{E}_{g_2,T}\big[Y^2(T)\big] = \mathcal{E}_{g_2,\sigma_0}\big[Y^2(\sigma_0)\big].$$

Combining with the hypothesis (H2) of Theorem 3.1, we have

$$\mathcal{E}_{g_2,\sigma_0}\big[Y^1(\sigma_0)\big] = \mathcal{E}_{g_1,\sigma_0}\big[Y^1(\sigma_0)\big] = \mathcal{E}_{g_1,\tau_0}[y_0] = \mathcal{E}_{g_2,\tau_0}[y_0] = \mathcal{E}_{g_2,\sigma_0}\big[Y^2(\sigma_0)\big]. \tag{4}$$

We define $\bar{g}_2(t,y,z)=g_2(t,y,z)\mathbf{1}_{[0,\sigma_0]}(t)$, for each $t\in[0,T]$, $(y,z)\in\mathbf{R}\times\mathbf{R}^d$, where $\mathbf{1}_{[0,\sigma_0]}$ denotes the indicator function of $[0,\sigma_0]$. Then, obviously $\mathcal{E}_{g_2,\sigma_0}[Y^i(\sigma_0)]=\mathcal{E}_{\bar{g}_2,T}[Y^i(\sigma_0)]$, $i=1,2$ (see [5], Proposition 2.5 and its proof for details). Since by Proposition 3.2 we have:

$$Y^1(\sigma_0)\geqslant Y^2(\sigma_0)\quad\text{and}\quad P\big(\big\{Y^1(\sigma_0)>Y^2(\sigma_0)\big\}\big)=P\big(\{\tau_0<\sigma_0\}\big)>0,$$

it follows from the strict comparison theorem (see Lemma 2.2(ii)) that

$$\mathcal{E}_{g_2,\sigma_0}\big(Y^1(\sigma_0)\big)=\mathcal{E}_{\bar{g}_2,T}\big(Y^1(\sigma_0)\big)>\mathcal{E}_{\bar{g}_2,T}\big(Y^2(\sigma_0)\big)=\mathcal{E}_{g_2,\sigma_0}\big(Y^2(\sigma_0)\big). \tag{5}$$

Clearly (5) is a contradiction to (4). Here we complete the proof of Theorem 3.1. □

Remark 2. Theorem 3.1 also holds if we replace (A5) by (A6).

Remark 3. Proposition 2.3 is a consequence of Theorem 3.1; let us sketch the proof:
For each stopping time $\tau\leqslant T$, let $\xi\in L^2(\Omega,\mathcal{F}_\tau,P)$, then $\xi\in L^2(\Omega,\mathcal{F}_T,P)$. Thanks to assumption (A4), we have $g_i(t,y,0)=0$ and hence we can verify that $\mathcal{E}_{g_i,\tau}[\xi]=\mathcal{E}_{g_i,T}[\xi]$, $i=1,2$. Thus under the assumption (A4), hypothesis (H1) $\Rightarrow$ (H2), so the proof is complete.

If we assume that g_1, g_2 satisfy assumptions (A1) and (A3), we can get another interesting result.

Theorem 3.3. *Let two generators* g_1, g_2 *satisfy assumptions* (A1), (A3) *and* (A6). *We assume moreover that* g_1, g_2 *satisfy hypothesis* (H3):

(H3) $\mathcal{E}_{g_1,r}[\xi]=\mathcal{E}_{g_2,r}[\xi]$, $\forall r\in[0,T]$, $\xi\in L^2(\Omega,\mathcal{F}_r,P)$.

Then for each triplet $(t,y,z)\in[0,T]\times\mathbf{R}\times\mathbf{R}^d$, *we have*:

$$P\text{-}a.s.,\quad g_1(t,y,z)=g_2(t,y,z).$$

Before we give the proof of the Theorem 3.3, we first prove the following proposition:

Proposition 3.4. *Let two generators* g_1, g_2 *satisfy assumptions* (A1) *and* (A3) *and hypothesis* (H3). *Let* $0\leqslant r\leqslant s\leqslant T$. *Then for* $\forall\xi\in L^2(\Omega,\mathcal{F}_s,P)$, P-*a.s., we have*:

$$\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]=\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r].$$

Proof. For $\forall\xi\in L^2(\Omega,\mathcal{F}_s,P)$, $A\in\mathcal{F}_r$, let $\mathbf{1}_A$ denote the indicator function of A. Thanks to assumption (A3): $g_i(t,0,0)\equiv 0$, we can see clearly that $\mathbf{1}_A g(t,y,z)=g(t,\mathbf{1}_A y,\mathbf{1}_A z)$. Therefore we can prove that

$$P\text{-a.s.},\quad \mathcal{E}_{g_i,s}[\mathbf{1}_A\xi|\mathcal{F}_r]=\mathbf{1}_A\mathcal{E}_{g_i,s}[\xi|\mathcal{F}_r],\quad i=1,2. \tag{6}$$

Now we set $A_0:=\{\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]>\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]\}$, then obviously $A_0\in\mathcal{F}_r$. Thanks to *(6)*, Lemma 2.1 and (H3), we can get that

$$\begin{aligned}\mathcal{E}_{g_1,r}\big[\mathbf{1}_{A_0}\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]\big]&=\mathcal{E}_{g_1,r}\big[\mathcal{E}_{g_1,s}[\mathbf{1}_{A_0}\xi|\mathcal{F}_r]\big]=\mathcal{E}_{g_1,s}[\mathbf{1}_{A_0}\xi]=\mathcal{E}_{g_2,s}[\mathbf{1}_{A_0}\xi]\\&=\mathcal{E}_{g_2,r}\big[\mathcal{E}_{g_2,s}[\mathbf{1}_{A_0}\xi|\mathcal{F}_r]\big]=\mathcal{E}_{g_1,r}\big[\mathcal{E}_{g_2,s}[\mathbf{1}_{A_0}\xi|\mathcal{F}_r]\big]=\mathcal{E}_{g_1,r}\big[\mathbf{1}_{A_0}\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]\big].\end{aligned}$$

From the definition of A_0, we have $\mathbf{1}_{A_0}\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]\geqslant\mathbf{1}_{A_0}\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]$. Therefore from the strict comparison theorem (see Lemma 2.2) we have: $P(A_0)=P(\{\mathbf{1}_{A_0}\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]>\mathbf{1}_{A_0}\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]\})=0$. Similarly we can prove that $P(\{\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]>\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]\})=0$. Thus P-a.s., $\mathcal{E}_{g_1,s}[\xi|\mathcal{F}_r]=\mathcal{E}_{g_2,s}[\xi|\mathcal{F}_r]$. Here we complete the proof of Proposition 3.4. □

Proof of Theorem 3.3. For each triplet $(t,y,z)\in[0,T[\times\mathbf{R}\times\mathbf{R}^d$, thanks to Briand, Coquet, Hu, Mémin and Peng [1, Proposition 2.3], we understand that L^2-$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}[\mathcal{E}_{g_i,t+\varepsilon}[y+z\cdot(B_{t+\varepsilon}-B_t)|\mathcal{F}_t]-y]=g_i(t,y,z)$, $i=1,2$. Thus there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$, such that P-a.s., $g_i(t,y,z)=\lim_{k\to\infty}n_k[\mathcal{E}_{g_i,t+1/n_k}[y+z\cdot(B_{t+1/n_k}-B_t)|\mathcal{F}_t]-y]$, $i=1,2$. Thanks to Proposition 3.4, we understand that for large enough k, P-a.s., $\mathcal{E}_{g_1,t+1/n_k}[y+z\cdot(B_{t+1/n_k}-B_t)|\mathcal{F}_t]=\mathcal{E}_{g_2,t+1/n_k}[y+z\cdot(B_{t+1/n_k}-B_t)|\mathcal{F}_t]$. Thus for the given triplet $(t,y,z)\in[0,T[\times\mathbf{R}\times\mathbf{R}^d$, we have: P-a.s., $g_1(t,y,z)=g_2(t,y,z)$. For the triplet $(T,y,z)\in\{T\}\times\mathbf{R}\times\mathbf{R}^d$, thanks to $(A6)$, we know that for pair (y,z), $g_i(t,y,z)$ is left continuous in T and thus we also have: P-a.s., $g_1(T,y,z)=\lim_{n\to\infty}g_1(T-1/n,y,z)=\lim_{n\to\infty}g_2(T-1/n,y,z)=g_2(T,y,z)$. Thus we complete the proof of Theorem 3.3. □

Remark 4. Proposition 2.3 can also be seen as a consequence of Theorem 3.3. Indeed, under the assumptions (A1) and (A4), we can verify that $\mathcal{E}_{g_i,r}[\xi]=\mathcal{E}_{g_i,T}[\xi]$, for $\forall r\in[0,T]$, $\xi\in L^2(\Omega,\mathcal{F}_r,P)$. Thus under the assumptions (A1) and (A4), hypothesis (H1) ⇒ (H3).

In Theorem 3.3 or in Proposition 2.3, if we assume g_1,g_2 only satisfy assumptions (A1), (A3) and (A5) and hypothesis (H1), can we prove that $g_1=g_2$? Generally the answer is negative.

Counterexample 3.1. Let us define two functions $h_1(t):=t$, $h_2(t):=T-t$, $\forall t\in[0,T]$. Define $g_1(t,y,z):=yh_1(t)$, $g_2(t,y,z):=yh_2(t)$, $\forall(t,y,z)\in[0,T]\times\mathbf{R}\times\mathbf{R}^d$. Then generators g_1,g_2 satisfy assumptions (A1), (A3) and (A5), but g_1,g_2 do not satisfy (A4).

Let $\xi\in L^2(\Omega,\mathcal{F}_T,P)$. Applying Itô's formula to $\mathcal{E}_{g_i,T}[\xi|\mathcal{F}_t]\exp(\int_0^t h_i(s)\,\mathrm{d}s)$, $i=1,2$, we can get that

$$\mathcal{E}_{g_i,T}[\xi|\mathcal{F}_t]=\left[\exp\left(\int_t^T h_i(s)\,\mathrm{d}s\right)\right]\mathbf{E}[\xi|\mathcal{F}_t],\quad t\in[0,T];\quad \mathcal{E}_{g_1,T}[\xi]=\exp\left(\frac{1}{2}T^2\right)\mathbf{E}[\xi]=\mathcal{E}_{g_2,T}[\xi].$$

For each $r\in\,]0,T[$, $\eta\in L^2(\Omega,\mathcal{F}_r,P)$, similarly we can obtain that

$$\mathcal{E}_{g_1,r}[\eta]=\left[\exp\left(\int_0^r h_1(s)\,\mathrm{d}s\right)\right]\mathbf{E}[\eta],\quad \mathcal{E}_{g_2,r}[\eta]=\left[\exp\left(\int_0^r h_2(s)\,\mathrm{d}s\right)\right]\mathbf{E}[\eta].$$

Hence g_1,g_2 satisfy (H1), but g_1,g_2 do not satisfy (H3). Obviously $g_1\neq g_2$.

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